Polarization and depolarization of light in optical scattering: polarimetric indices of purity

José J. Gil

Workshop
Light scattering from microstructures
Laredo , Spain (11/09/1998)

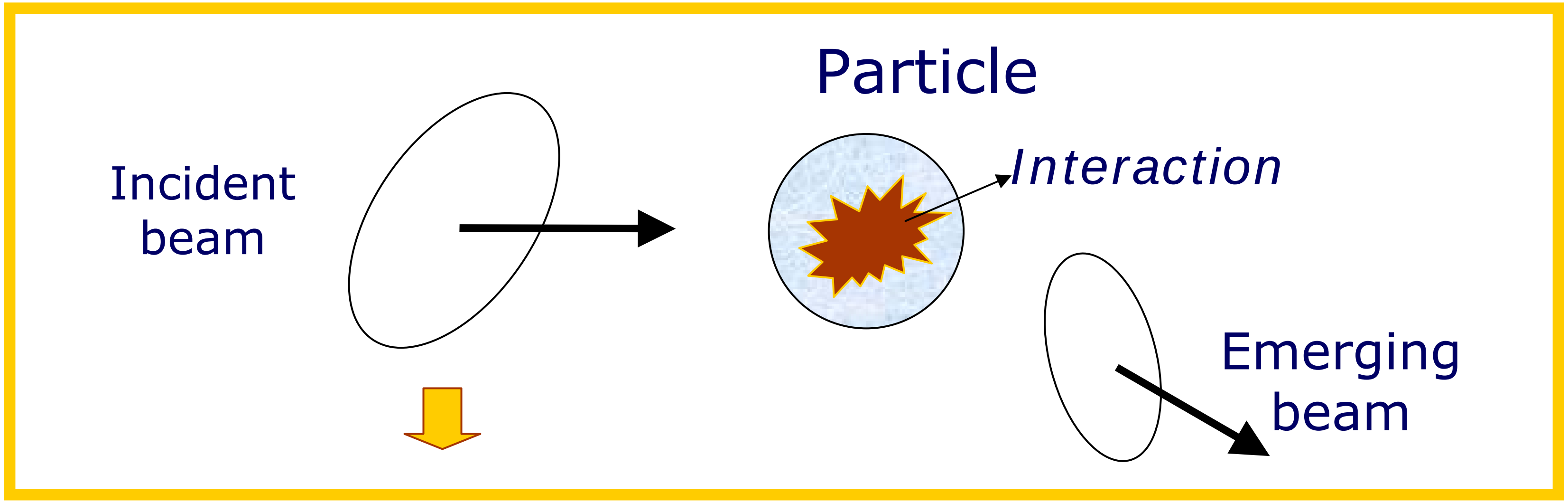
Outline

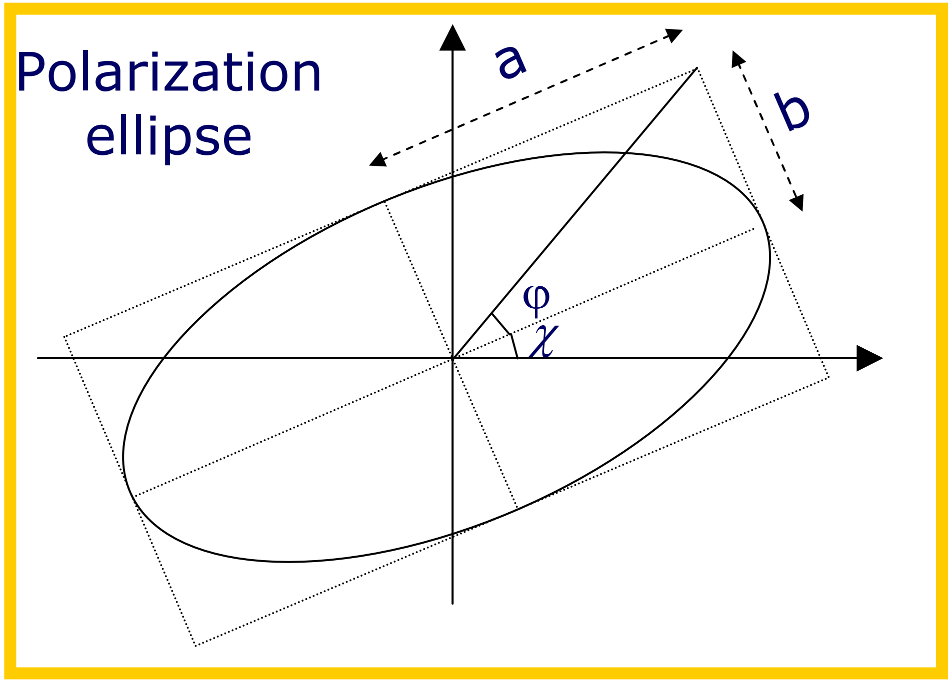
- Questions
- Basic interaction
- Macroscopic interaction
- General structure of Mueller matrices
- Parallel decomposition of Mueller matrices
- Degree of purity
- Indices of purity

Questions

- How many non-null elements are in the Mueller matrix?
- How many independent elements are in the Mueller matrix?
- How can we extract the maximum physical information from these parameters?
- How many parameters characterize completely the depolarization properties?

Basic interaction: Transformation of the polarization ellipse

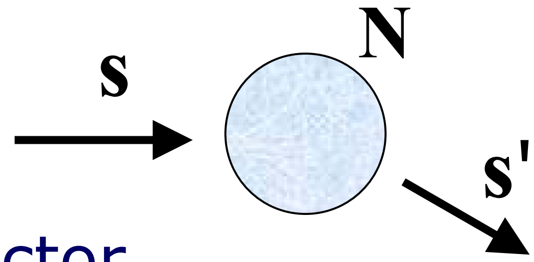




- *Azimuth* → χ
- *Ellipticity* → φ
- *Size* → I
- *“Stability”* → P

J. J. Gil. Workshop: Light scattering from microstructures, Laredo, Spain (11/09/1998)

Basic interaction: Mathematical description



Stokes vector

$$\mathbf{s} \equiv \begin{bmatrix} I \\ IP\cos 2\varphi \cos 2\chi \\ IP\cos 2\varphi \sin 2\chi \\ IP \sin 2\varphi \end{bmatrix}$$

- *I intensity*
- P *degree of polarization*
- χ *azimuth*
- φ *ellipticity*

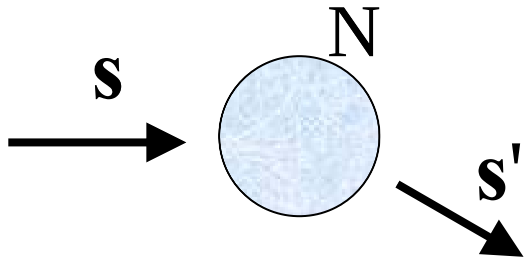
$$\mathbf{s}' = \mathbf{N}\,\mathbf{s}$$

Mueller-Jones matrix

- *Incident beam: P=1*
- *Single interaction* ⇒ *P'=1*

J. J. Gil. Workshop: Light scattering from microstructures, Laredo, Spain (11/09/1998)

Basic interaction: Mueller matrix N



when

Single interaction or *Coherent superposition of emerging beams*

"Pure case": N is a Mueller-Jones matrix

$$\mathbf{N} = \mathbf{A}\left(\mathbf{T} \otimes \mathbf{T}^{+}\right)\mathbf{A}^{-1}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{bmatrix}$$

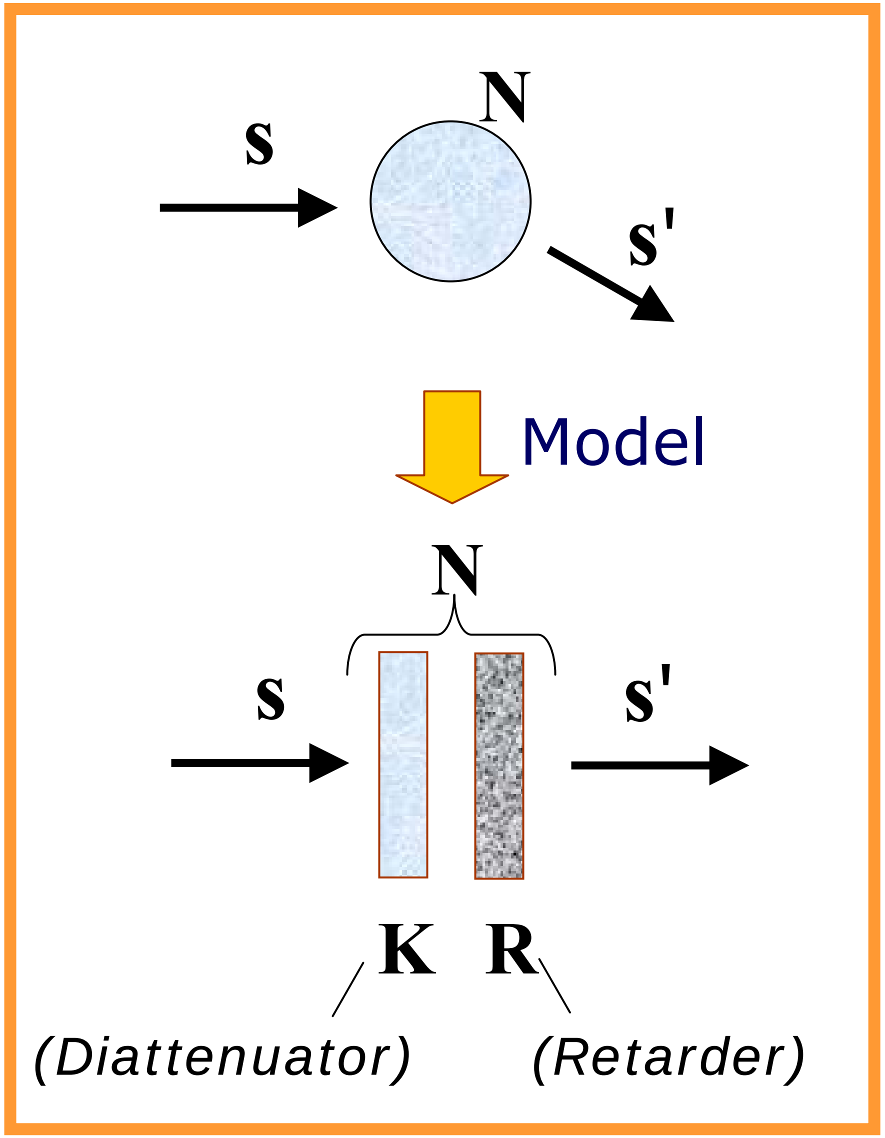
J. J. Gil. Workshop: Light scattering from microstructures, Laredo, Spain (11/09/1998)

Polar decomposition of N

$$\mathbf{N} = \mathbf{RK}$$

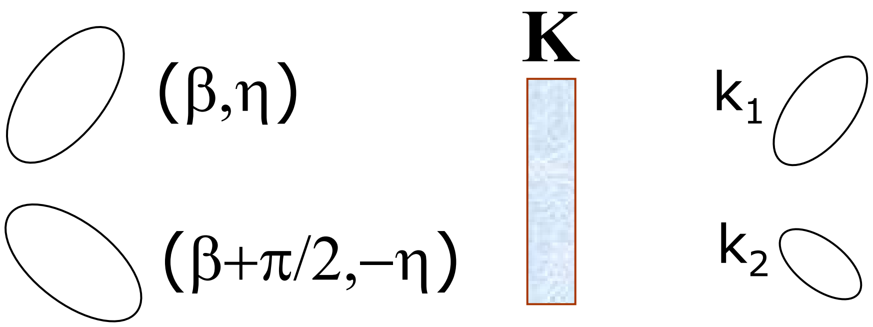
R: elliptical retarder
(3 parameters)

K: elliptical partial polarizer
(4 parameters)



Equivalent system: physical parameters

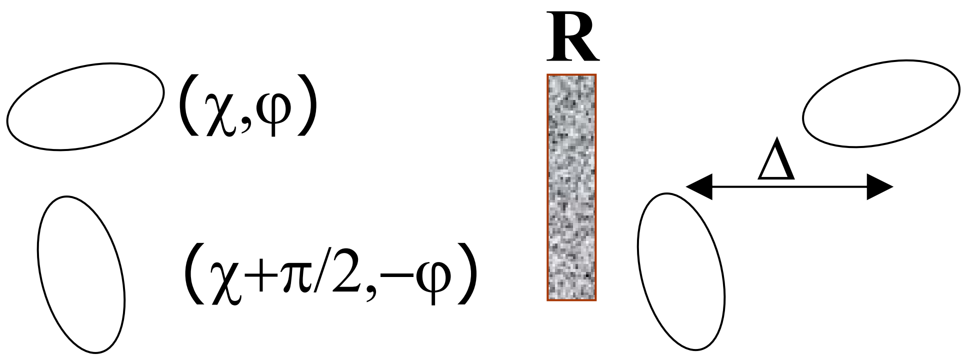
Diattenuator (Elliptical partial polarizer)



Parameters

- *Eigenstates* (β,η)
- *Transmittances* k_1, k_2

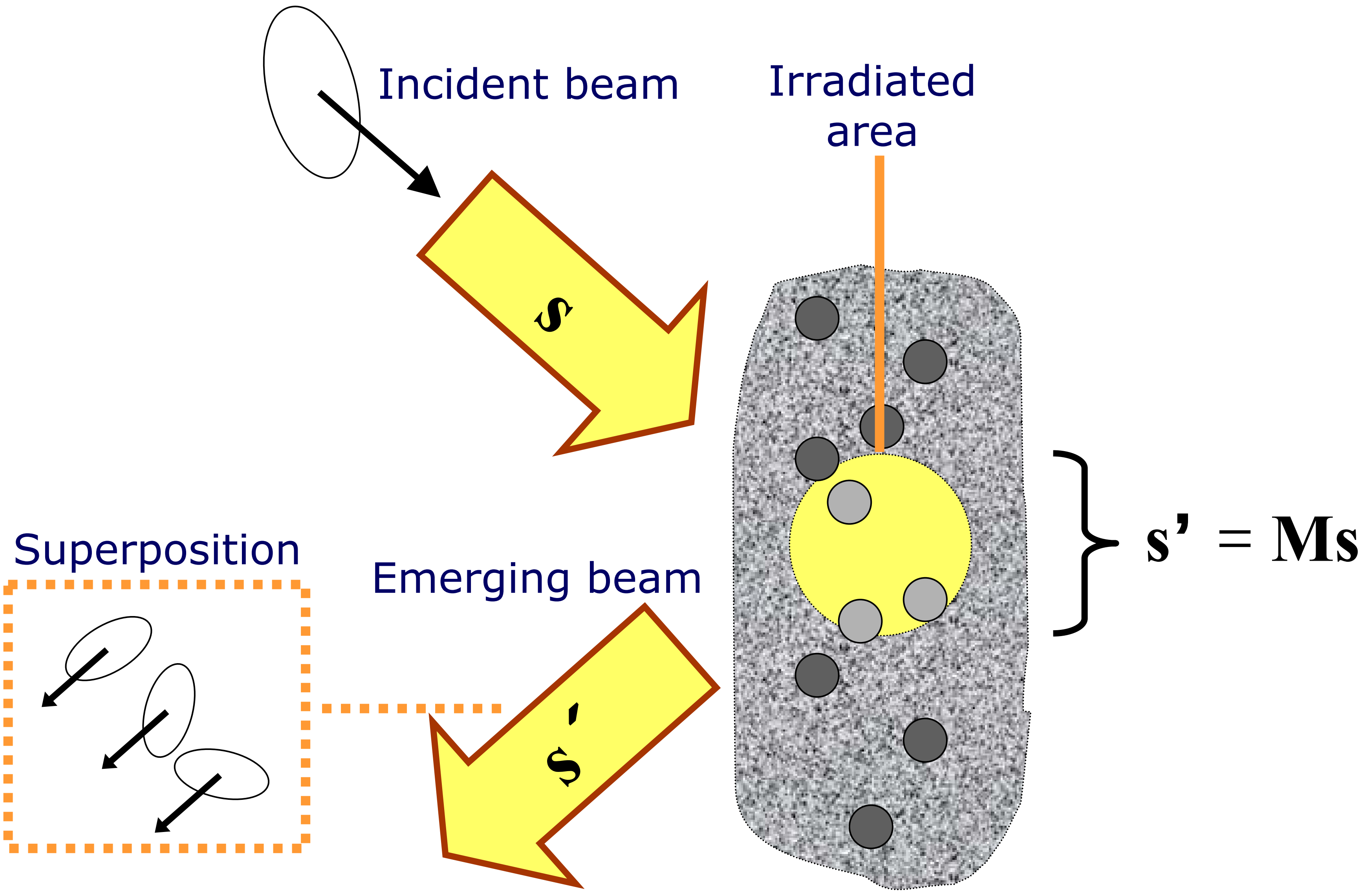
Elliptical retarder



Parameters

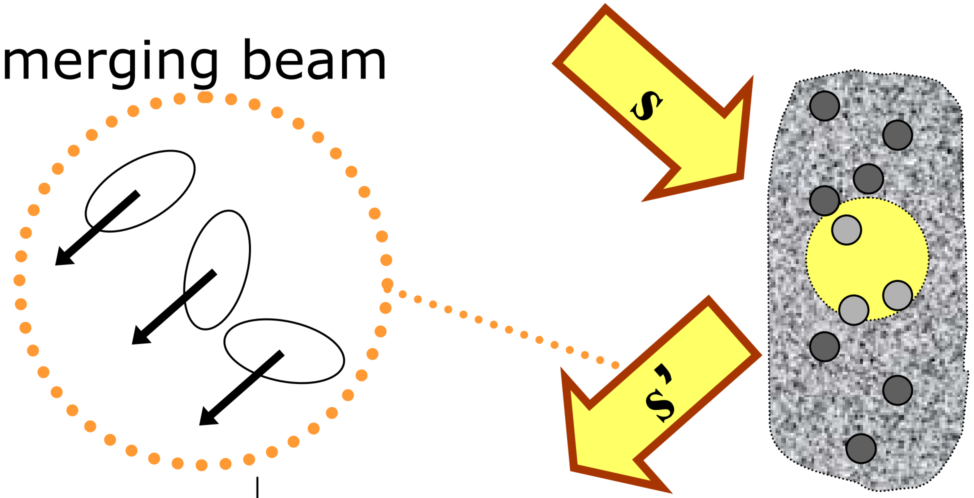
- *Eigenstates* (χ,φ)
- *Retardation* Δ

Macroscopic interaction



J. J. Gil. Workshop: Light scattering from microstructures, Laredo, Spain (11/09/1998)

Macroscopic interaction



$$p_i = \frac{I^{(i)}}{I}$$

$$\sum_i p_i = 1$$

Coherent superposition of elementary emerging beams	*Partially coherent superposition of elementary emerging beams*
⇩	⇩
"Pure case"	"General case"
$\mathbf{M} \equiv \mathbf{N} = \mathbf{R}(\chi,\varphi,\Delta)\,\mathbf{K}(k_1, k_2, \beta,\eta)$	$\mathbf{M} = \sum_i p_i \mathbf{N}^{(i)}$

J. J. Gil. Workshop: Light scattering from microstructures, Laredo, Spain (11/09/1998)

Mueller matrix as a parallel composition of pure Mueller matrices. Coherency matrix

$$\mathbf{M} = \sum_i p_i \mathbf{N}^{(i)}$$

$$\mathbf{H} = \frac{1}{4}\begin{pmatrix} \begin{matrix} m_{00}+m_{01} \\ +m_{10}+m_{11} \end{matrix} & \begin{matrix} m_{02}+m_{12} \\ +i\left(m_{03}+m_{13}\right) \end{matrix} & \begin{matrix} m_{20}+m_{21} \\ -i\left(m_{30}+m_{31}\right) \end{matrix} & \begin{matrix} m_{22}+m_{33} \\ +i\left(m_{23}-m_{32}\right) \end{matrix} \\ \\ \begin{matrix} m_{02}+m_{12} \\ -i\left(m_{03}+m_{13}\right) \end{matrix} & \begin{matrix} m_{00}-m_{01} \\ +m_{10}-m_{11} \end{matrix} & \begin{matrix} m_{22}-m_{33} \\ -i\left(m_{23}+m_{32}\right) \end{matrix} & \begin{matrix} m_{20}-m_{21} \\ -i\left(m_{30}-m_{31}\right) \end{matrix} \\ \\ \begin{matrix} m_{20}+m_{21} \\ +i\left(m_{30}+m_{31}\right) \end{matrix} & \begin{matrix} m_{22}-m_{33} \\ +i\left(m_{23}+m_{32}\right) \end{matrix} & \begin{matrix} m_{00}+m_{01} \\ -m_{10}-m_{11} \end{matrix} & \begin{matrix} m_{02}-m_{12} \\ +i\left(m_{03}-m_{13}\right) \end{matrix} \\ \\ \begin{matrix} m_{22}+m_{33} \\ -i\left(m_{23}-m_{32}\right) \end{matrix} & \begin{matrix} m_{20}-m_{21} \\ +i\left(m_{30}-m_{31}\right) \end{matrix} & \begin{matrix} m_{02}-m_{12} \\ -i\left(m_{03}-m_{13}\right) \end{matrix} & \begin{matrix} m_{00}-m_{01} \\ -m_{10}+m_{11} \end{matrix} \end{pmatrix}$$

General structure of a Mueller matrix

- **H**: Hermitian positive semi-definite, 4x4 complex matrix
- **H**: Covariance matrix (four zero-mean complex variables)

$$h_{kl} \equiv \left\langle t_k t_l^* \right\rangle_e$$

$$k,l = 0,1,2,3$$

$$\mathbf{H} = \begin{bmatrix} \sigma_0^2 & \mu_{01}\sigma_0\sigma_1 & \mu_{02}\sigma_0\sigma_2 & \mu_{03}\sigma_0\sigma_3 \\ \mu_{01}^*\sigma_0\sigma_1 & \sigma_1^2 & \mu_{12}\sigma_1\sigma_2 & \mu_{13}\sigma_1\sigma_3 \\ \mu_{02}^*\sigma_0\sigma_2 & \mu_{12}^*\sigma_1\sigma_2 & \sigma_2^2 & \mu_{23}\sigma_2\sigma_3 \\ \mu_{03}^*\sigma_0\sigma_3 & \mu_{13}^*\sigma_1\sigma_3 & \mu_{23}^*\sigma_2\sigma_3 & \sigma_3^2 \end{bmatrix}$$

General characterization of Mueller matrices

Eigenvalue conditions (4):

- *The four eigenvalues of **H** are non-negative*

or

- *Four nested principal minors of **H** are non-negative*

Transmittance conditions (2)

1. Forward transmittance condition

$$m_{00} + \left(m_{01}^2 + m_{02}^2 + m_{03}^2 \right)^{1/2} \leq 1$$

2. Reverse transmittance condition (derived from the nature of a Mueller matrix as a parallel composition of pure Mueller matrices)

$$m_{00} + \left(m_{10}^2 + m_{20}^2 + m_{30}^2 \right)^{1/2} \leq 1$$

J. J. Gil. Workshop: Light scattering from microstructures, Laredo, Spain (11/09/1998)

Statistical nature of **H**

$$h_{kl} \equiv \mu_{kl} \sigma_k \sigma_l$$

- σ_k^2 Variances
- μ_{kl} Complex correlation coefficients

$$\rho_{kl} \equiv |\mu_{kl}|, \quad \beta_{kl} \equiv \arg(\mu_{kl})$$

Statistical nature of **M**

$$\mathbf{M} = \begin{bmatrix} \sigma_0^2+\sigma_1^2+\sigma_2^2+\sigma_3^2 & \sigma_0^2-\sigma_1^2-\sigma_2^2+\sigma_3^2 & \begin{matrix}2\rho_{02}\sigma_0\sigma_2 c_{02}\\ +2\rho_{13}\sigma_1\sigma_3 c_{13}\end{matrix} & \begin{matrix}2\rho_{02}\sigma_0\sigma_2 s_{02}\\ -2\rho_{13}\sigma_1\sigma_3 s_{13}\end{matrix} \\ \sigma_0^2-\sigma_1^2+\sigma_2^2-\sigma_3^2 & \sigma_0^2+\sigma_1^2-\sigma_2^2-\sigma_3^2 & \begin{matrix}2\rho_{02}\sigma_0\sigma_2 c_{02}\\ -2\rho_{13}\sigma_1\sigma_3 c_{13}\end{matrix} & \begin{matrix}2\rho_{02}\sigma_0\sigma_2 s_{02}\\ +2\rho_{13}\sigma_1\sigma_3 s_{13}\end{matrix} \\ \begin{matrix}2\rho_{03}\sigma_0\sigma_3 c_{03}\\ +2\rho_{12}\sigma_1\sigma_2 c_{12}\end{matrix} & \begin{matrix}2\rho_{03}\sigma_0\sigma_3 c_{03}\\ -2\rho_{12}\sigma_1\sigma_2 c_{12}\end{matrix} & \begin{matrix}2\rho_{01}\sigma_0\sigma_1 c_{01}\\ +2\rho_{23}\sigma_2\sigma_3 c_{23}\end{matrix} & \begin{matrix}-2\rho_{01}\sigma_0\sigma_1 s_{01}\\ -2\rho_{23}\sigma_2\sigma_3 s_{23}\end{matrix} \\ \begin{matrix}-2\rho_{03}\sigma_0\sigma_3 s_{03}\\ +2\rho_{12}\sigma_1\sigma_2 s_{12}\end{matrix} & \begin{matrix}-2\rho_{03}\sigma_0\sigma_3 s_{03}\\ -2\rho_{12}\sigma_1\sigma_2 s_{12}\end{matrix} & \begin{matrix}2\rho_{01}\sigma_0\sigma_1 s_{01}\\ -2\rho_{23}\sigma_2\sigma_3 s_{23}\end{matrix} & \begin{matrix}2\rho_{01}\sigma_0\sigma_1 c_{01}\\ -2\rho_{23}\sigma_2\sigma_3 c_{23}\end{matrix} \end{bmatrix}$$

$$s_{kl} \equiv \sin(\beta_{kl})$$
$$c_{kl} \equiv \cos(\beta_{kl})$$

J. J. Gil. Workshop: Light scattering from microstructures, Laredo, Spain (11/09/1998)

Canonical expansion of the coherency matrix

H *represents univocally the Mueller matrix and vice-versa and that this relation can be written as follows*

$$\mathbf{H} = \frac{1}{4}\sum_{k=0}^{3}\sum_{l=0}^{3} m_{kl}\mathbf{E}_{kl}$$

$\mathbf{E}_{kl}$ (k,l = 0, 1, 2, 3): set of 16 modified Dirac matrices

$$\mathbf{E}_{kl} \equiv \boldsymbol{\sigma}_k \otimes \boldsymbol{\sigma}_l$$

These coefficients are 16 directly measurable quantities (observable parameters), i. e. the 16 elements of the Mueller matrix $\mathbf{M}$ associated with $\mathbf{H}$

J. J. Gil. Workshop: Light scattering from microstructures, Laredo, Spain (11/09/1998)

Canonical (spectral) expansion of the coherency matrix

$$\mathbf{H} = \frac{1}{4}\sum_{k=0}^{3}\sum_{l=0}^{3} m_{kl}\mathbf{E}_{kl}$$

This expansion provides a fundamental significance for the elements of the Mueller matrix, in addition to their phenomenological significance.

The relation between the covariance matrix $\mathbf{H}$ and its corresponding Mueller matrix $\mathbf{M}$ is, in fact, analogous to that between the polarization matrix and its corresponding Stokes parameters

J. J. Gil. Workshop: Light scattering from microstructures, Laredo, Spain (11/09/1998)

Incoherent components of a Mueller matrix

- It is possible to classify the Mueller Matrices according to the rank of $\mathbf{H}$, i. e. according to the number of nonzero eigenvalues
- For example, $\mathbf{H}$ corresponds to a Mueller-Jones matrix $\mathbf{N}$ when only one eigenvalue is nonzero
- As $\mathbf{H}$ is Hermitian, it can be diagonalized through a similarity transformation, given by a unitary matrix $\mathbf{W}$, such as

$$\mathbf{H} = \mathbf{W}\,\mathbf{\Lambda}\,\mathbf{W}^{+}$$

Λ is the eigenvalue diagonal matrix $\mathbf{D}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$

Incoherent components of a Mueller matrix

Then, we can write **H** as

$$\mathbf{H} = \mathbf{W\Lambda W}^{+} = \sum_{k=0}^{3} \lambda_k \mathbf{\Lambda}_k =$$

$$= \lambda_0 \mathbf{WD}(1,0,0,0)\mathbf{W}^{+} + \lambda_1 \mathbf{WD}(0,1,0,0)\mathbf{W}^{+}$$

$$+\lambda_2 \mathbf{WD}(0,0,1,0)\mathbf{W}^{+} + \lambda_3 \mathbf{WD}(0,0,0,1)\mathbf{W}^{+}$$

⇓

$$\mathbf{M} = \frac{\lambda_0}{m_{00}}\mathbf{N}_0 + \frac{\lambda_1}{m_{00}}\mathbf{N}_1 + \frac{\lambda_2}{m_{00}}\mathbf{N}_2 + \frac{\lambda_3}{m_{00}}\mathbf{N}_3 \ , \qquad \mathbf{N}_k \equiv \mathbf{N}(\mathbf{\Lambda}_k)$$

Parallel decomposition of a Mueller matrix

$$\mathbf{M} = \frac{\lambda_0}{m_{00}}\mathbf{N}_0 + \frac{\lambda_1}{m_{00}}\mathbf{N}_1 + \frac{\lambda_2}{m_{00}}\mathbf{N}_2 + \frac{\lambda_3}{m_{00}}\mathbf{N}_3$$

This decomposition provides a physical interpretation analogous to the decomposition of a polarization matrix into a superposition of two orthogonal-polarized, incoherent, linearly independent beams whose intensities are the two eigenvalues of the polarization matrix.

We see that the system can be considered a parallel combination of one to four pure elements

Purity criterion

A necessary and sufficient condition for a (physical) Mueller matrix $\mathbf{M}$ to be a Mueller-Jones matrix is that $\mathbf{M}$ satisfies the equality

$$\operatorname{tr}\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right) = 4m_{00}^{2}$$

Degree of purity

The purity degree can be defined as

$$\mathrm{G}(\mathbf{M}) = \sqrt{\frac{\mathrm{tr}\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right) - m_{00}^2}{3m_{00}^2}}$$

$$0 \le \mathrm{G}(\mathbf{M}) \le 1$$

$$\mathrm{G}(\mathbf{H}) = \sqrt{\frac{4\,\mathrm{tr}\left(\mathbf{H}^2\right) - \left(\mathrm{tr}\mathbf{H}\right)^2}{3\left(\mathrm{tr}\mathbf{H}\right)^2}}$$

J. J. Gil. Workshop: Light scattering from microstructures, Laredo, Spain (11/09/1998)

Indices of purity

$$G_1 \equiv \frac{\lambda_0 - \lambda_1}{m_{00}}$$

$$G_2 \equiv \frac{(\lambda_0 + \lambda_1) - (\lambda_2 + \lambda_3)}{m_{00}}$$

$$G_3 \equiv \frac{\lambda_2 - \lambda_3}{m_{00}}$$

Analogy between the indices of purity and the degree of polarization

$$P \equiv \frac{\lambda_0' - \lambda_1'}{I} \quad \longrightarrow$$

$$G_1 \equiv \frac{\lambda_0 - \lambda_1}{m_{00}}$$

$$G_2 \equiv \frac{(\lambda_0 + \lambda_1) - (\lambda_2 + \lambda_3)}{m_{00}}$$

$$G_3 \equiv \frac{\lambda_2 - \lambda_3}{m_{00}}$$

λ_0', λ_1' Eigenvalues of the polarization matrix

I, Intensity

P, Degree of polarization

The indices of purity constitute a minimum set of invariant parameters that characterize completely the depolarization properties of the system

J. J. Gil. Workshop: Light scattering from microstructures, Laredo, Spain (11/09/1998)

Indices of purity and depolarization

$$G^2 = \frac{1}{3}\left(2G_1^2 + G_2^2 + 2G_3^2\right)$$

The degree of purity G is the modulus of the "depolarization vector"

$$\mathbf{G} \equiv \frac{1}{\sqrt{3}}\left(\sqrt{2}G_1\,, G_2\,, \sqrt{2}G_3\right)$$

J. J. Gil. Workshop: Light scattering from microstructures, Laredo, Spain (11/09/1998)