

Polarimetry applied to scattering measurements

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Workshop

Light scattering from microstructures

Laredo , Spain (11/09/1998)

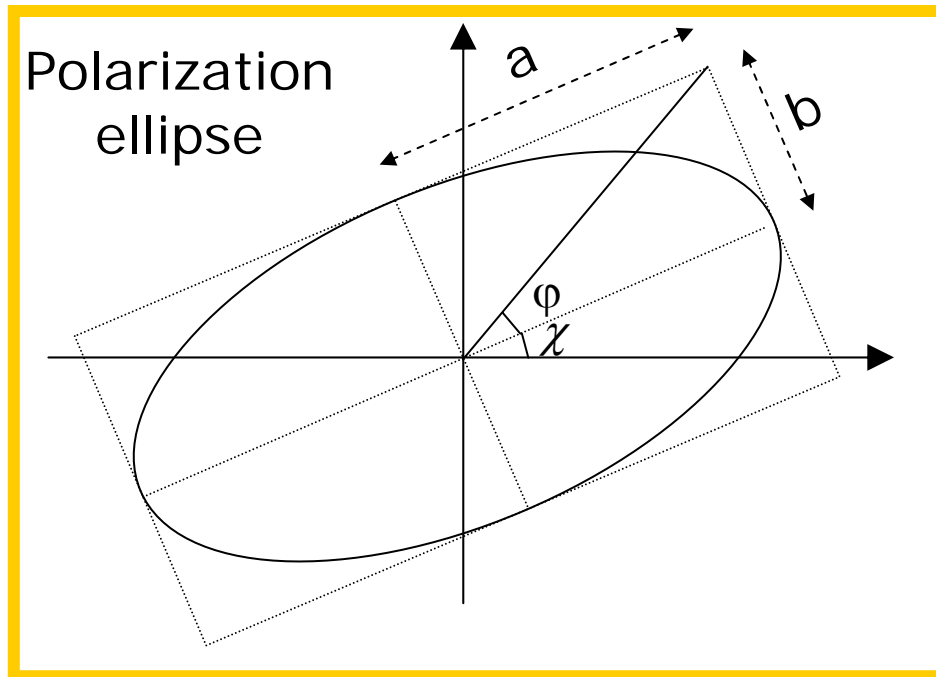
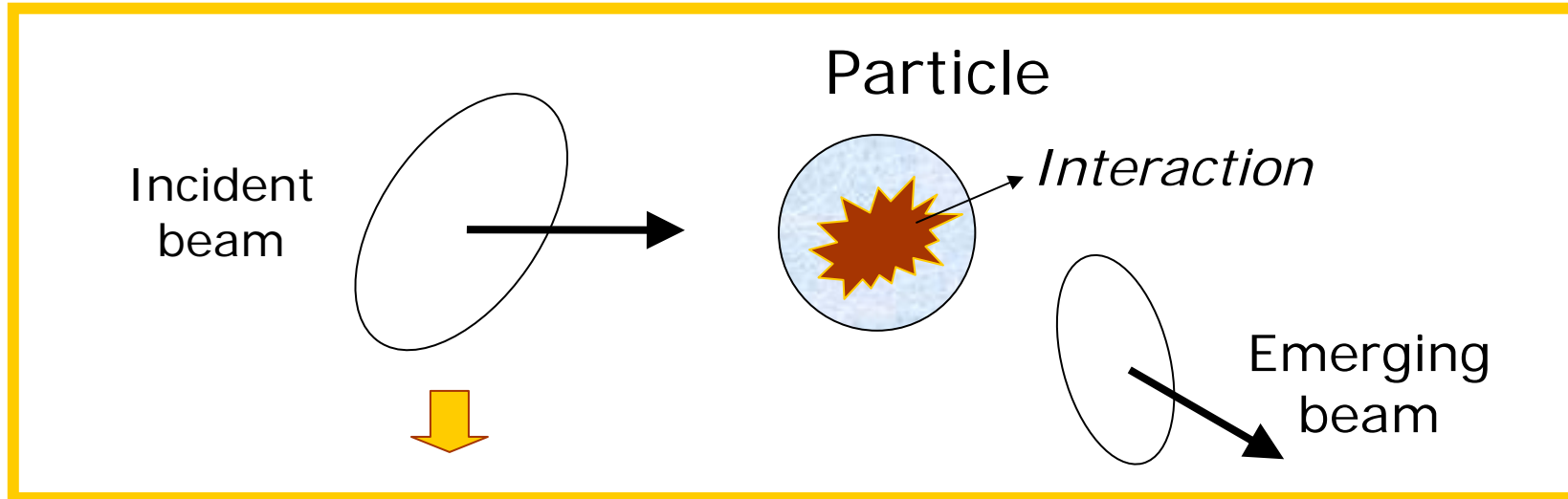
Outline

- Questions
- Basic interaction
- Macroscopic interaction
- General structure of Mueller matrices
- Parallel decomposition of Mueller matrices
- Degree of purity
- Indices of purity

Questions

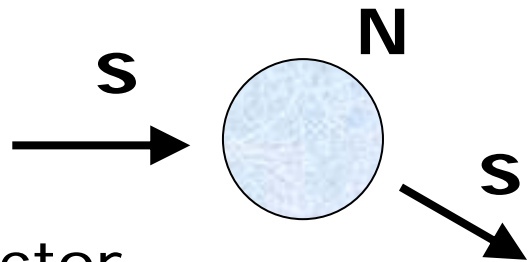
- ¿How many non-null elements are in the Mueller matrix?
- ¿How many independent elements are in the Mueller matrix?
- ¿How can we extract the maximum physical information from these parameters?

Basic interaction: Transformation of the polarization ellipse



- *Azimuth* $\rightarrow \chi$
- *Ellipticity* $\rightarrow \varphi$
- *Size* $\rightarrow I$
- *"Stability"* $\rightarrow P$

Basic interaction: Mathematical description



Stokes vector

$$\mathbf{s} \equiv \begin{bmatrix} I \\ IP \cos 2\varphi \cos 2\chi \\ IP \cos 2\varphi \sin 2\chi \\ IP \sin 2\varphi \end{bmatrix}$$

- I intensity
- P degree of polarization
- χ azimuth
- φ ellipticity

$$\mathbf{s}' = \mathbf{N} \mathbf{s}$$

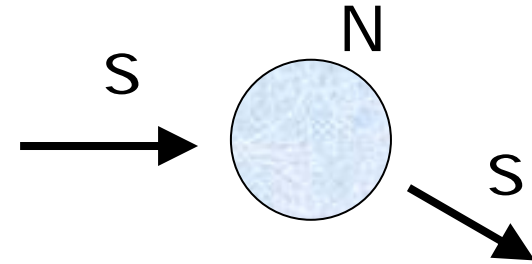


Mueller matrix

• Incident beam: $P=1$

• Single interaction $\rightarrow P'=1$

Basic interaction: Mueller matrix N



when

Single interaction

or

Coherent superposition of emerging beams

“Pure case”: N is a Mueller-Jones matrix

$$N = A(T \otimes T^+)A^{-1}$$

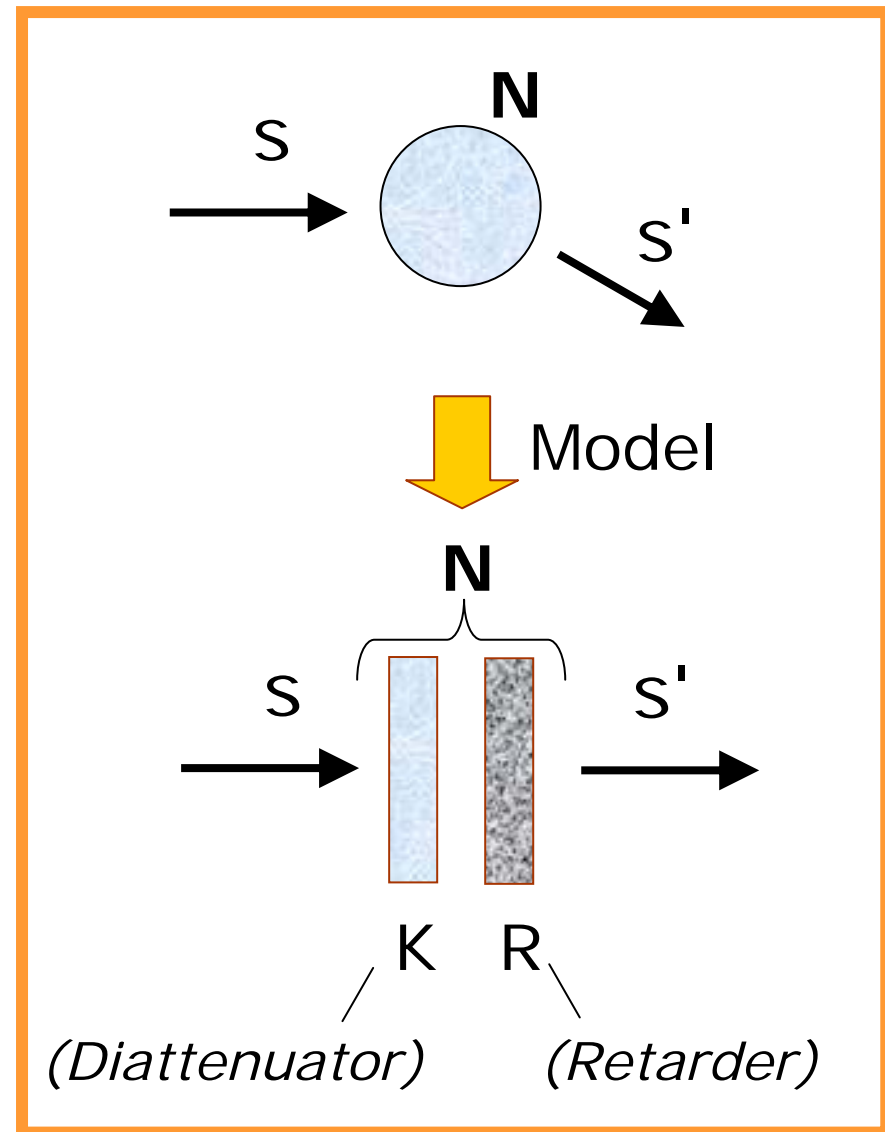
$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{bmatrix}$$

Polar decomposition of N

$$\mathbf{N} = \mathbf{R}\mathbf{K}$$

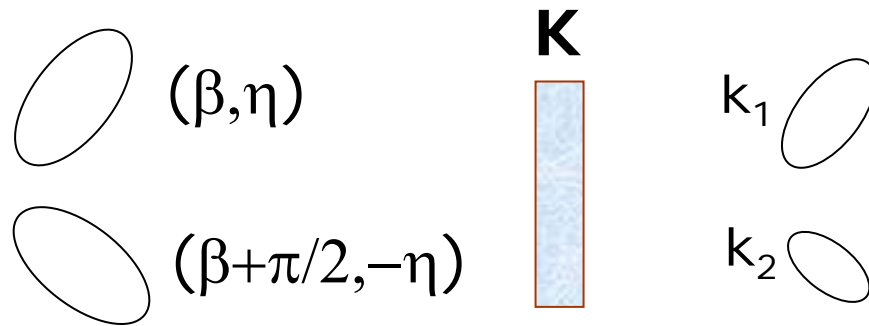
R: elliptical retarder
(3 parameters)

K: elliptical partial
polarizer
(4 parameters)



Equivalent system: physical parameters

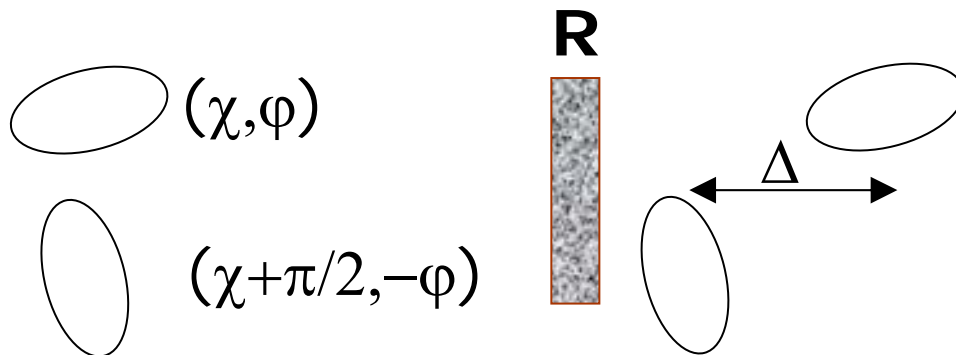
Diattenuator (Elliptical partial polarizer)



Parameters

- *Eigenstates* (β, η)
- *Transmittances* k_1, k_2

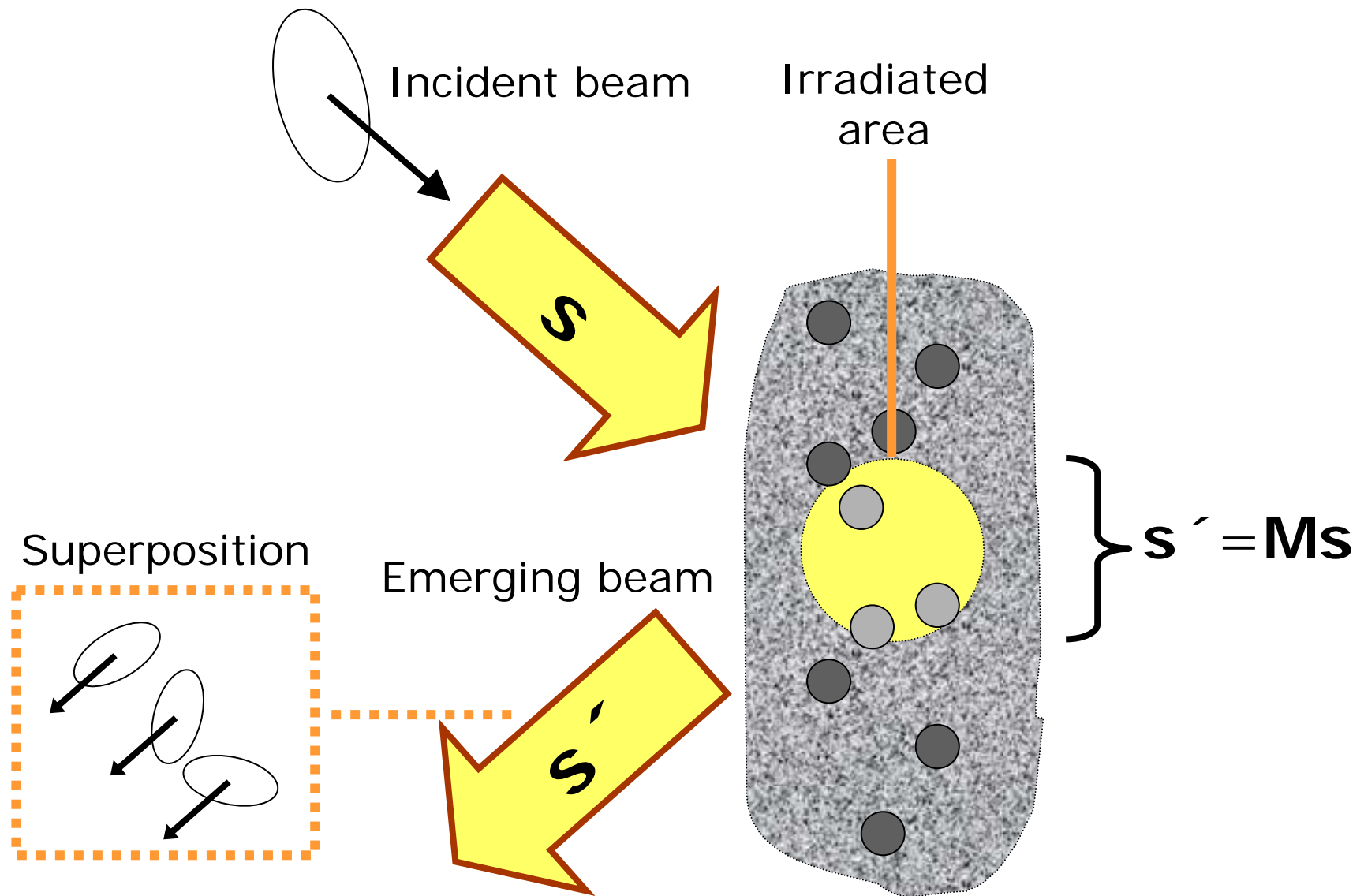
Elliptical retarder



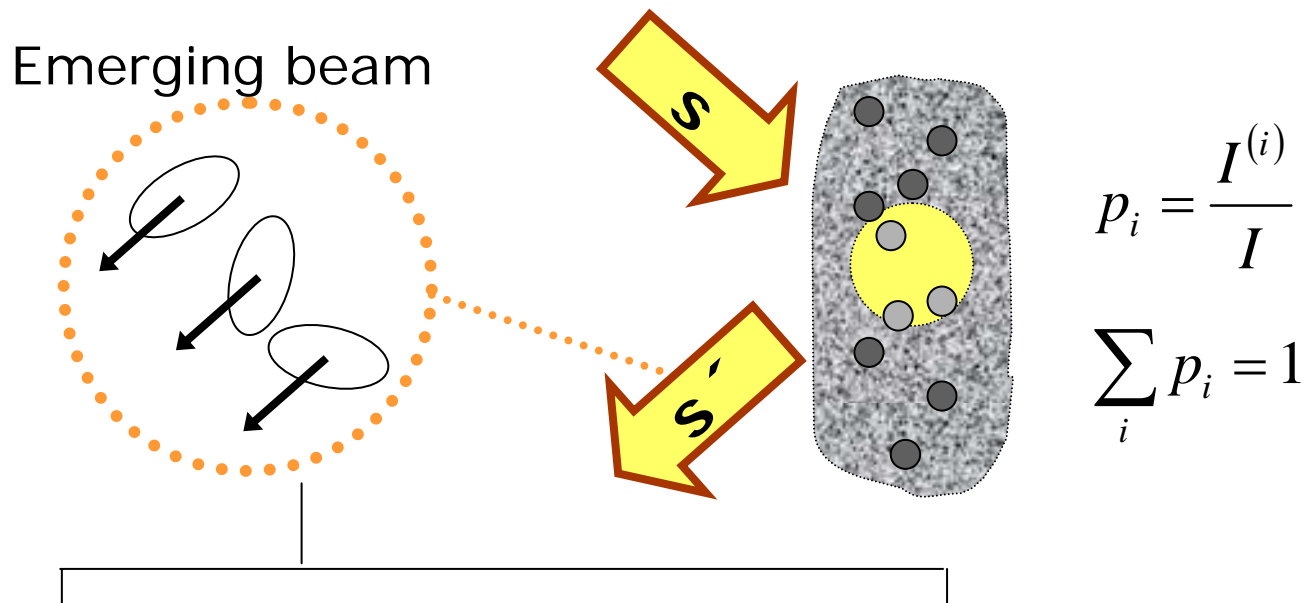
Parameters

- *Eigenstates* (χ, φ)
- *Retardation* Δ

Macroscopic interaction



Macroscopic interaction



Coherent superposition of elementary emerging beams



"Pure case"

$$\mathbf{M} \equiv \mathbf{N} = \mathbf{R}(\chi, \varphi, \Delta) \mathbf{K}(k_1, k_2, \beta, \eta)$$

Partially coherent superposition of elementary emerging beams



"General case"

$$\mathbf{M} = \sum_i p_i \mathbf{N}^{(i)}$$

Mueller matrix as a parallel composition of pure Mueller matrices. Coherency matrix

$$\mathbf{M} = \sum_i p_i \mathbf{N}^{(i)}$$

$$\mathbf{H} = \frac{1}{4} \begin{pmatrix} m_{00} + m_{01} & m_{02} + m_{12} & m_{20} + m_{21} & m_{22} + m_{33} \\ +m_{10} + m_{11} & +i(m_{03} + m_{13}) & -i(m_{30} + m_{31}) & +i(m_{23} - m_{32}) \\ m_{02} + m_{12} & m_{00} - m_{01} & m_{22} - m_{33} & m_{20} - m_{21} \\ -i(m_{03} + m_{13}) & +m_{10} - m_{11} & -i(m_{23} + m_{32}) & -i(m_{30} - m_{31}) \\ m_{20} + m_{21} & m_{22} - m_{33} & m_{00} + m_{01} & m_{02} - m_{12} \\ +i(m_{30} + m_{31}) & +i(m_{23} + m_{32}) & -m_{10} - m_{11} & +i(m_{03} - m_{13}) \\ m_{22} + m_{33} & m_{20} - m_{21} & m_{02} - m_{12} & m_{00} - m_{01} \\ -i(m_{23} - m_{32}) & +i(m_{30} - m_{31}) & -i(m_{03} - m_{13}) & -m_{10} + m_{11} \end{pmatrix}$$

General structure of a Mueller matrix

- **H**: Hermitian positive semi-definite, 4x4 complex matrix
- **H**: Covariance matrix (four zero-mean complex variables)

$$h_{kl} \equiv \left\langle t_k t_l^* \right\rangle_e$$

$$k, l = 0, 1, 2, 3$$

$$\mathbf{H} = \begin{bmatrix} \sigma_0^2 & \mu_{01} \sigma_0 \sigma_1 & \mu_{02} \sigma_0 \sigma_2 & \mu_{03} \sigma_0 \sigma_3 \\ \mu_{01}^* \sigma_0 \sigma_1 & \sigma_1^2 & \mu_{12} \sigma_1 \sigma_2 & \mu_{13} \sigma_1 \sigma_3 \\ \mu_{02}^* \sigma_0 \sigma_2 & \mu_{12}^* \sigma_1 \sigma_2 & \sigma_2^2 & \mu_{23} \sigma_2 \sigma_3 \\ \mu_{03}^* \sigma_0 \sigma_3 & \mu_{13}^* \sigma_1 \sigma_3 & \mu_{23}^* \sigma_2 \sigma_3 & \sigma_3^2 \end{bmatrix}$$

General characterization of Mueller matrices

Eigenvalue conditions (4):

- *The four eigenvalues of \mathbf{H} are non-negative*
- or
- *Four nested principal minors of \mathbf{H} are non-negative*

Transmittance conditions (2)

1. *Forward transmittance ^{λ} condition*

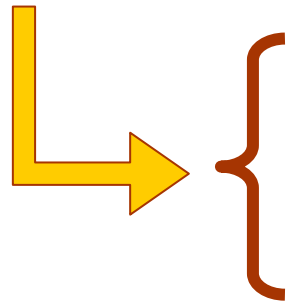
$$m_{00} + \left(m_{01}^2 + m_{02}^2 + m_{03}^2 \right)^{1/2} \leq 1$$

2. *Reverse transmittance condition (derived from the nature of a Mueller matrix as a parallel composition of pure Mueller matrices)*

$$m_{00} + \left(m_{10}^2 + m_{20}^2 + m_{30}^2 \right)^{1/2} \leq 1$$

Statistical nature of \mathbf{H}

$$h_{kl} \equiv \mu_{kl} \sigma_k \sigma_l$$



σ_k^2 Variances

μ_{kl} Complex correlation coefficients

$$\rho_{kl} \equiv |\mu_{kl}|, \quad \beta_{kl} \equiv \arg(\mu_{kl})$$

Statistical nature of \mathbf{M}

$$\mathbf{M} = \begin{bmatrix} \sigma_0^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2 & \sigma_0^2 - \sigma_1^2 - \sigma_2^2 + \sigma_3^2 & 2\rho_{02}\sigma_0\sigma_2c_{02} & 2\rho_{02}\sigma_0\sigma_2s_{02} \\ \sigma_0^2 - \sigma_1^2 + \sigma_2^2 - \sigma_3^2 & \sigma_0^2 + \sigma_1^2 - \sigma_2^2 - \sigma_3^2 & +2\rho_{13}\sigma_1\sigma_3c_{13} & -2\rho_{13}\sigma_1\sigma_3s_{13} \\ 2\rho_{03}\sigma_0\sigma_3c_{03} & 2\rho_{03}\sigma_0\sigma_3c_{03} & 2\rho_{01}\sigma_0\sigma_1c_{01} & -2\rho_{01}\sigma_0\sigma_1s_{01} \\ +2\rho_{12}\sigma_1\sigma_2c_{12} & -2\rho_{12}\sigma_1\sigma_2c_{12} & +2\rho_{23}\sigma_2\sigma_3c_{23} & -2\rho_{23}\sigma_2\sigma_3s_{23} \\ -2\rho_{03}\sigma_0\sigma_3s_{03} & -2\rho_{03}\sigma_0\sigma_3s_{03} & 2\rho_{01}\sigma_0\sigma_1s_{01} & 2\rho_{01}\sigma_0\sigma_1c_{01} \\ +2\rho_{12}\sigma_1\sigma_2s_{12} & -2\rho_{12}\sigma_1\sigma_2s_{12} & -2\rho_{23}\sigma_2\sigma_3s_{23} & -2\rho_{23}\sigma_2\sigma_3c_{23} \end{bmatrix}$$

$$S_{kl} \equiv \sin(\beta_{kl})$$

$$C_{kl} \equiv \cos(\beta_{kl})$$

Canonical expansion of the coherency matrix

\mathbf{H} represents univocally the Mueller matrix and vice-versa and that this relation can be written as follows

$$\mathbf{H} = \frac{1}{4} \sum_{k=0}^3 \sum_{l=0}^3 m_{kl} \mathbf{E}_{kl}$$

\mathbf{E}_{kl} ($k, l = 0, 1, 2, 3$): set of 16 modified Dirac matrices

$$\mathbf{E}_{kl} \equiv \boldsymbol{\sigma}_k \otimes \boldsymbol{\sigma}_l$$

These coefficients are 16 directly measurable quantities (observable parameters), i. e. the 16 elements of the Mueller matrix \mathbf{M} associated with \mathbf{H}

Canonical (spectral) expansion of the coherency matrix

$$\mathbf{H} = \frac{1}{4} \sum_{k=0}^3 \sum_{l=0}^3 m_{kl} \mathbf{E}_{kl}$$

This expansion provides a fundamental significance for the elements of the Mueller matrix, in addition to their phenomenological significance. The relation between the covariance matrix \mathbf{H} and its corresponding Mueller matrix is, in fact, analogous to that between the polarization matrix and its corresponding Stokes parameters

Incoherent components of a Mueller matrix

- It is possible to classify the Mueller Matrices according to the rank of \mathbf{H} , i. e. according to the number of nonzero eigenvalues
- For example, \mathbf{H} corresponds to a Mueller-Jones matrix \mathbf{N} when only one eigenvalue is nonzero
- As \mathbf{H} is Hermitian, it can be diagonalized through a similarity transformation, given by a unitary matrix \mathbf{W} , such as

$$\mathbf{H} = \mathbf{W} \mathbf{\Lambda} \mathbf{W}^+$$

$\mathbf{\Lambda}$ is the eigenvalue diagonal matrix $\mathbf{D}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$

Incoherent components of a Mueller matrix

Then, we can write \mathbf{H} as

$$\begin{aligned}\mathbf{H} &= \mathbf{W}\mathbf{\Lambda}\mathbf{W}^+ = \sum_{k=0}^3 \lambda_k \mathbf{\Lambda}_k = \\ &= \lambda_0 \mathbf{W}\mathbf{D}(1,0,0,0) \mathbf{W}^+ + \lambda_1 \mathbf{W}\mathbf{D}(0,1,0,0) \mathbf{W}^+ \\ &+ \lambda_2 \mathbf{W}\mathbf{D}(0,0,1,0) \mathbf{W}^+ + \lambda_3 \mathbf{W}\mathbf{D}(0,0,0,1) \mathbf{W}^+\end{aligned}$$



$$\mathbf{M} = \frac{\lambda_0}{m_{00}} \mathbf{N}_0 + \frac{\lambda_1}{m_{00}} \mathbf{N}_1 + \frac{\lambda_2}{m_{00}} \mathbf{N}_2 + \frac{\lambda_3}{m_{00}} \mathbf{N}_3, \quad \mathbf{N}_k \equiv \mathbf{N}(\mathbf{\Lambda}_k)$$

Parallel decomposition of a Mueller matrix

$$\mathbf{M} = \frac{\lambda_0}{m_{00}} \mathbf{N}_0 + \frac{\lambda_1}{m_{00}} \mathbf{N}_1 + \frac{\lambda_2}{m_{00}} \mathbf{N}_2 + \frac{\lambda_3}{m_{00}} \mathbf{N}_3$$

This decomposition provides a physical interpretation analogous to the decomposition of a polarization matrix into a superposition of two orthogonal-polarized, incoherent, linearly independent beams whose intensities are the two eigenvalues of the polarization matrix.

We see that the system can be considered a parallel combination of one to four pure elements

Purity criterion

A necessary and sufficient condition for a (physical) Mueller matrix \mathbf{M} to be a Mueller-Jones matrix is that \mathbf{M} satisfies the equality

$$\text{tr}(\mathbf{M}^T \mathbf{M}) = 4m_{00}^2$$

Degree of purity

The purity degree can be defined as

$$\mathbf{G}(\mathbf{M}) = \sqrt{\frac{\text{tr}(\mathbf{M}^T \mathbf{M}) - m_{00}^2}{3m_{00}^2}}$$

$$0 \leq \mathbf{G}(\mathbf{M}) \leq 1$$

$$\mathbf{G}(\mathbf{H}) = \sqrt{\frac{4 \text{tr}(\mathbf{H}^2) - (\text{tr}\mathbf{H})^2}{3(\text{tr}\mathbf{H})^2}}$$

Indices of purity

$$G_1 \equiv \frac{\lambda_0 - \lambda_1}{m_{00}}$$

$$G_2 \equiv \frac{(\lambda_0 + \lambda_1) - (\lambda_2 + \lambda_3)}{m_{00}}$$

$$G_3 \equiv \frac{\lambda_2 - \lambda_3}{m_{00}}$$

$$G^2 = \frac{1}{3} \left(2G_1^2 + G_2^2 + 2G_3^2 \right)$$