University of Zaragoza

Department of Optics

Determination of polarization parameters in matricial representation

Theoretical contribution and development of an automatic measurement device

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Index

I Introduction 6

II. Formalisms of representation of polarized light and optical media 9

II.1 Electric field vector and polarization ellipse 10

II.2 Jones calculus 13

II.3 Stokes-Mueller formalism 15

II.4 Coherency matrix and coherency vector formalisms 20

II.4.1 N-type optical media 22

II.4.2. G-type optical media 22

II.5 Relations between the different formalisms 25

II.5.1 Some formal considerations 25

II.5.2 Relations about the characterization of light 26

II.5.3 Relations about the characterization of optical media 28

II.6. Optical media. Notation 34

II.6.1. Partial polarizer 34

II.6.2. Total polarizers 37

II.6.3. Ideal retarders 38

II.6.4. Non-ideal retarders 39

II.6.5. SL(2C) group and Lorentz group 41

II.7 Polar decomposition 42

II.8. Theorems 42

III. Properties of the matrices that represent optical media 46

III.1. Degree of polarization in the different formalisms 47

III.2. Construction of a generic Mueller matrix 48

III.2.1 Equivalence general theorem ... 48

III.2.2. Polar decomposition theorem ... 50

III.3. Classification of the N-type Mueller matrices ... 53

III.4. Restrictive relations in a Mueller matrix ... 59

III.5. Restrictive relations in a Matrix V in the CVF formalism ... 69

III.6. Norm condition in Mueller matrices ... 71

III.7. Norm condition in Jones matrices ... 75

III.8. Norm condition in matrices V of the formalism CVF. ... 77

III.9. Indices of polarization and depolarization. ... 78

IV. Method for the dynamic determination of Mueller matrices ... 85

IV.1. Measurement device ... 86

IV.2. Calibration ... 96

IV.3 Apparatus signal ... 101

IV.4. Computerized Fourier analysis of the recorded signal ... 103

V. Dynamic method for the analysis of polarized light ... 104

V.I. Analysis device ... 105

V.2. Calibration ... 107

V.3. Sensitivity of the apparatus signal with respect to the calibration parameters ... 109

VI. Experimental device ... 111

VI.1. General experimental assembly ... 112

VI.2 Possible causes of errors in the measurements ... 117

VI.3. Computerized data processing ... 119

VI.3.1. MAPAR program ... 119

VI.3.2. MEREL program ... 120

VI.3.3. MEPOL and STOKES programs ... 121

VII. Calibration and some results 122

VII.1. Determination of Stokes parameters 123

VII.1.1. Calibration 123

VII.1.2. Elliptical polarization 124

VII.2. Determination of Mueller matrices 125

VII.2.1. Calibration 125

VII.2.2. Commercial retarder 130

VII.2.3. Commercial linear polarizer 132

VII.2.4. System of two linear retarders. 133

VII.2.5. System with three linear retarders 134

VII.2.6. System composed of a polarizer and a retarder. 136

VII.3 Discussion 139

Conclusions 141

Chapter I

Introduction

The analysis of polarized light and the determination of characteristic optical parameters of media with respect to the polarization phenomena represent a work area in Optics that has had an impact on several physical subjects and in other sciences. Nevertheless, a renewed interest in this area has been recently appreciated, and a new and plentiful literature has arisen. In the field of the ellipsometry it is worth mentioning the works of R.M.A. Azzam and N.M. Bashara, mainly developed on the basis of the of R.C. Jones [1] mathematical formalism, that produced a complete treatise [2] with both, the latest dynamic and static techniques to determine optical parameters and concrete application for thin films study, blood coagulum formation, etc. P.S. Theocaris and E.E. Gdoutos [3] have presented a rigorous study of the photoelasticity matricial theory in which R.C. Jones and H. Mueller formalism is used. The work of H.C. Van de Hulst [4] is also worth mentioning, because of the study of light dispersion by means of particles with several forms and sizes, giving matricial models in the Mueller formalism for each case. This work has recently prompt to diverse studies about natural atmosphere, artificial aerosols, marine hydrosols, etc [5].

Other related subjects of interest that are being developed nowadays are the study of birefringent or dichroic spectral filters [6], natural rotatory power [7], birefringence in optical fibers [8], etc.

The behavior of optical media that are active to the polarization can be studied by means of the Stokes-Mueller's formalism [4,9,10,11,12]. However, the elements of the Mueller matrix associated with an optical medium don't include direct information about the relevant parameters of the physical behavior of the medium, so it is necessary to do a previous study to classify the Mueller matrices according to the corresponding properties of the optical medium in order to facilitate the extraction of the parameters with direct physical interpretation. Several authors like K.D. Abhyankar and A.L. Fymat [13], R. Barakat [14], E.S. Fry and G.W. Kattawar [15], and R.W. Schaefer [16] have studied the relation among the elements of any Mueller matrix. Furthermore, R.C. Jones [1], J.R. Priebe [17] and C.Whitney [18] have established several theorems where the polarimetric equivalence between complex and simple systems with few optical media is demonstrated. Moreover, in the latest years there has been a tendency to the development of dynamic devices for the determination of optical parameters [19]. So, P.S. Hauge and H. Dill [20] have presented a dynamic method for the analysis of the polarized light. The work of Hauge and Dill work, together with the paper of E. Collet [21], in which the polarized light that emerges from a device with a rotatory retarder is considered, were the basis of our previous work, presented as Degree Thesis [22], where an experimental device with two rotatory retarders and two fixed linear polarizer was developed. Later, other authors have presented different devices that use non ideal rotatory retarders, like P.S. Hauge´s [23] one, or electro-optic modulator as the built by R.C. Thomson, J.B. Bottiger and E.S. Fry [24].

The aim of this work has been the development of a dynamic method, as general as possible, for the determination of Mueller matrices and the analysis of polarized light. This method has to admit a self-calibration, avoiding the use of tests and patterns during the calibrate operation and, thus, the problems related to the set-up, which generate instrumental limitations and systematic errors that are difficult to identify within the experimental results.

In our work we have considered the contribution of numerous authors, who use different formalisms for the representation and treatment of polarimetric properties of light and media. This fact has prompt us to include in this memory a chapter dedicated to the presentation and interpretation of the different formalisms, analyzing the relations among them and including the most important theorems relative to the equivalence and reciprocity of the optical systems. This Chapter II tries to give self-sufficiency to the memory and, although it includes a summary of the theoretical framework, it also contains some original contributions.

In Chapter III the relations between the elements of a generic Mueller matrix and the optical parameters that are characteristics of different equivalent systems are obtained and analyzed. Likewise, we analyze in detail the restrictive relations among the elements of the Mueller matrix, justifying them on physical bases and interpreting them in the framework of other formalisms. All of this has allowed us to establish a theorem that is useful to distinguish nondepolarizing systems from depolarizing ones, and furthermore, we have defined a set of parameters that indicate the degree of polarization and the degree of depolarization, introduced by any optical medium.

In Chapter IV, we present our dynamic method for the determination of Mueller matrices. It is based on the Fourier analysis of the intensity signal of the light emerging from a system with two rotating non-ideal linear retarders, where the optical medium whose Mueller matrix is to be measured is placed in the middle of them. The measurement device has several optical components, whose characteristic parameters are determined by means of a calibration operation. This calibration has the peculiarity of being made with the signal generated by the device, without any optical medium used like test or pattern.

The particularization of this measurement method for the analysis of polarized light is included in Chapter V, together with a calibration method.

In Chapter VI we describe the experimental measurement device, developed and designed by us, analyzing the main effects that can be sources of errors in the measurements.

In Chapter VII, the results corresponding with the self-calibration of our experimental device are presented, allowing us to estimate its accuracy. Furthermore, the results corresponding to several optical systems under measurement are presented. These results are analyzed with the help of the relations and theorems given in chapters II and III, and they are also used for illustrating the behavior of our experimental device.

Chapter II

Formalisms of representation of polarized light and optical media

According to the electromagnetic theory, light propagates in space by means of transverse electromagnetic waves, mathematically represented by the solutions of the Maxwell equations, which can be split as a sum of plane monochromatic waves.

The light vector is defined by means of the electric field vector. This vector is well defined for each particular type of totally polarized light and, thus, any polarized light can be described using the concepts of vectorial calculus. With this vectorial description any problem related with the propagation, refraction and reflection of polarized light through optical media can be managed. However, calculations are usually very complicated and make it difficult to solve the problems. This is the reason for introducing other descriptions for the polarized light. For each one of these descriptions, there is a matricial model that allows us to describe the optical properties of those material media that affect to the polarization of the light going through them. Hereafter, we will use the expression 'going through' to indicate the cases of transmission and reflection of light.

In general, light beams are polychromatic. A wave is said to be monochromatic when it only contains one discrete frequency with zero spectral width. An intermediate case is a quasi-monochromatic wave, characterized by a thin spectral line, with a very small, but nonzero width.

It is important to point out that the cases of totally polarized monochromatic and quasi-monochromatic light are polarimetrically indistinguishable [2], in any interaction phenomenon with optical media. This fact justifies the supposition of monochromaticity for totally polarized quasi-monochromatic waves.

II.1 Electric field vector and polarization ellipse

It is necessary to introduce a vectorial description of the polarized light in order to present a consistent and uniform notation along our work.

A uniform and plane monochromatic wave propagating in a homogenous and isotropic medium along Z axis in a Cartesian system of reference XYZ can be expressed in the form

$$\mathbf{E} = E_x \vec{\mathbf{i}} + E_y \vec{\mathbf{j}} \tag{II.1}$$

where **i**, **j** are unit vectors along X and Y directions, respectively, and the components of **E** are given by

$$E_x = A_x \cos\left(-\omega t + \frac{2\pi z}{\lambda} + \delta_x\right) = A_x \cos\left(\nu + \delta_x\right) \quad \text{(II.2.a)}$$

$$E_y = A_y \cos\left(-\omega t + \frac{2\pi z}{\lambda} + \delta_y\right) = A_y \cos\left(\nu + \delta_y\right) \quad \text{(II.2.b)}$$

with $\nu \equiv -\omega t + \frac{2\pi z}{\lambda}$; or in complex notation

$$E_x = A_x e^{i(\nu+\delta_x)} \quad \text{(II.3.a)}$$

$$E_y = A_y e^{i(\nu+\delta_y)} \quad \text{(II.3.b)}$$

knowing that the imaginary part of these expressions has not physical meaning.

The parameters A_x, A_y are the amplitudes according to the X and Y axes, λ is the wavelength, ω is the angular frequency; and δ_x, δ_y are phase constants.

From (II.3) and using some trigonometric relations, it is easy to demonstrate the following relation [3, 25]

$$\frac{E_x^2}{A_x^2} + \frac{E_y^2}{A_y^2} - 2\frac{E_x E_y}{A_x A_y}\cos\delta = \sin^2\delta \quad \text{(II.4.a)}$$

where

$$E_y = A_y e^{i(\nu+\delta_y)},\ \delta = (\delta_x - \delta_y) \quad \text{(II.4.b)}$$

Equation (II.4) represents an ellipse that is called *the polarization ellipse* (Fig. II.1). Its eccentricity and axes orientation on the plane XY depends on δ, but neither t nor Z.

Let α be the angle given by

$$\tan\alpha \equiv \frac{A_y}{A_x} \quad \text{(II.5)}$$

ψ the ellipticity of the polarization ellipse, and χ the azimuth of the major semi-axis of the ellipse with respect to the positive direction of the X axis. These angles are represented in Fig. II.2 and the following relations can be demonstrated [25, 26]

$$\tan 2\chi = \tan 2\alpha \cos\delta \quad \text{(II.6.a)}$$

$$\sin 2\psi = \sin 2\alpha \sin \delta \quad \text{(II.6.b)}$$

$$\cos 2\alpha = \cos 2\psi \cos 2\chi \quad \text{(II.6.c)}$$

$$\tan \delta = \frac{\tan 2\psi}{\sin 2\chi} \quad \text{(II.6.d)}$$

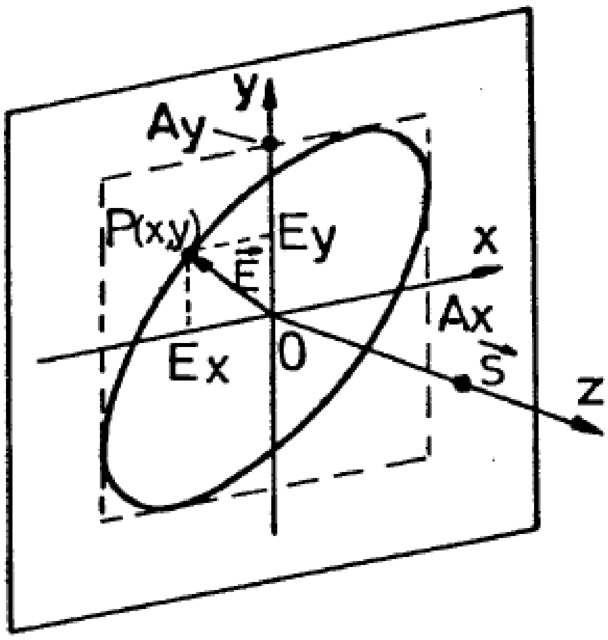


Fig. II.1 – Polarization ellipse

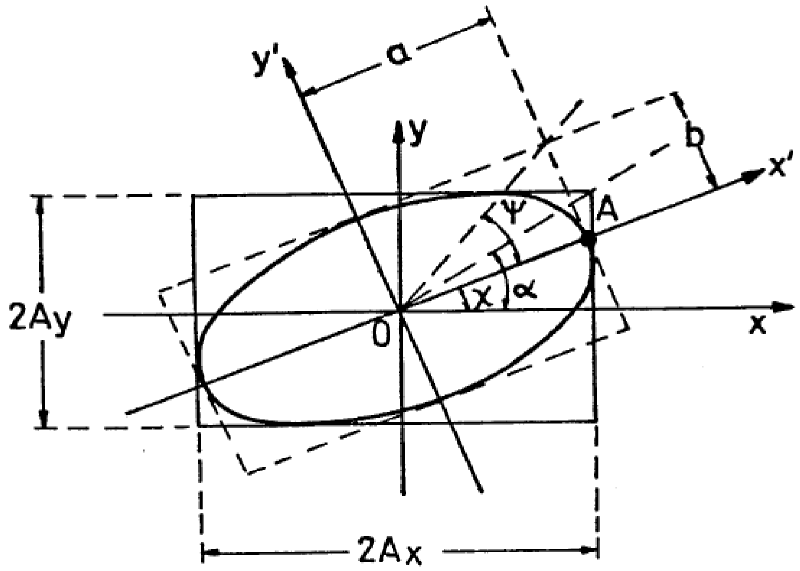


Fig. II.2 – Geometric representation of the parameters associated with the polarization ellipse

II.2 Jones calculus

A Jones vector is a column vector composed of two complex elements, namely the components E_x and E_y of the electric vector **E** of the light beam. The Jones vector, for the more general case of elliptical polarization, is defined as [1]

$$\boldsymbol{\varepsilon} \equiv \begin{pmatrix} E_x \\ E_y \end{pmatrix} = e^{iu} \begin{pmatrix} A_x e^{-i\delta/2} \\ A_y e^{i\delta/2} \end{pmatrix} \tag{II.7}$$

with

$$u = \upsilon + \frac{\delta_x + \delta_y}{2} \tag{II.8.a}$$

$$\delta = \delta_y - \delta_x \tag{II.8.b}$$

The amplitudes A_x, A_y and the relative phase δ are enough to define the polarization ellipse. However, the vector $\boldsymbol{\varepsilon}$ has information of the both phases, δ_x and δ_y, separately. This fact shows that, in general, a Jones vector is characterized by two independent complex numbers, i.e. by four real quantities.

There are problems in which the absolute phase is irrelevant. In such cases we can write the Jones vector as follows

$$\boldsymbol{\varepsilon} = \begin{pmatrix} A_x e^{-i\delta/2} \\ A_y e^{i\delta/2} \end{pmatrix} \tag{II.9}$$

In other cases, the Jones vector is normalized in such a way that the intensity value is unity $\boldsymbol{\varepsilon}^{+}\boldsymbol{\varepsilon} = A_x^2 + A_y^2 = 1$.

Two states of polarization with Jones vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ are called orthogonal when $\boldsymbol{\varepsilon}^{+}\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}'^{+}\boldsymbol{\varepsilon} = 0$. The orthogonal vectors correspond to polarization ellipses with the same ellipticity, opposite rotation directions and perpendicular major axes.

The coherent superposition of two beams of polarized light can be expressed as the sum of their corresponding Jones vectors.

When a monochromatic wave of light, as the one indicated in (II.1), passes through a linear optical medium that does not produce incoherent effects, the emerging wave is a linear transformation

$$\begin{aligned} E_x' &= A_1 E_x + A_3 E_y \\ E_y' &= A_4 E_x + A_2 E_y \end{aligned} \tag{II.10}$$

whereA_1, A_2, A_3, A_4 are complex coefficients that depend on the nature of the optical medium. Thus, the transformation (II.10) can be written as follows

$$\begin{pmatrix} E_x' \\ E_y' \end{pmatrix} = \begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \tag{II.11}$$

or

$$\boldsymbol{\varepsilon}' = \mathbf{J}\boldsymbol{\varepsilon} \tag{II.12}$$

J being the complex matrix defined by

$$\mathbf{J} = \begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \end{pmatrix} \tag{II.13}$$

Matrix **J** matrix is called the Jones matrix associated with the optical medium considered. The elements of **J** are usually written in two alternative ways

$$\mathbf{J} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} J_1 & J_3 \\ J_4 & J_2 \end{pmatrix} \tag{II.14}$$

Hereafter, when a matrix is represented with the **J** letter, it must be understood that it is a Jones matrix.

The Jones matrix associated with a succession of optical media can be obtained as the product of the matrices associated with the respective medium [2]. This is easily proved if we apply (II.12) successively.

From the definition of the Jones vector given in (II.7) we see that this vector is only defined for totally polarized light. This implies that we cannot represent with a Jones matrix a medium that reduces the degree of polarization of the light beam going through it.

The Jones matrix associated with a group of optical media, which are passed through in parallel by a coherent light beam, is given by the sum of the Jones matrices associated with these media.

The utility of the JCF* formalism is restricted to the problems related to totally polarized light.

Hereafter, we understand like "N type" those optical media that are represented by a Jones matrix, and "G type" those in general.

II.3 Stokes-Mueller formalism

Next we present a summary of the Stokes vector and Mueller matrices formalism, whose abbreviated name is SMF.

A Stokes vector is a column vector composed of four real elements S_0, S_1, S_2, S_3; in the case of a totally polarized light beam they are defined as follows [27]

* *For the sake of simplicity, we will use abbreviations to indicate the mathematical formalism. In this way, we use JCF to indicate the Jones calculus formalism.*

$$
\begin{aligned}
s_0 &= E_x E_x^* + E_y E_y^* \\
s_1 &= E_x E_x^* - E_y E_y^* \\
s_2 &= E_x E_y^* + E_y E_x^* \\
s_3 &= i\left(E_x E_y^* - E_y E_x^*\right)
\end{aligned}
\tag{II.15}
$$

where the complex notation for E_x and E_y has been adopted.

There is the following alternative way to write the Stokes parameters

$$
\begin{aligned}
s_0 &= A_x^2 + A_y^2 \\
s_1 &= A_x^2 - A_y^2 \\
s_2 &= 2A_x A_y \cos\delta \\
s_3 &= 2A_x A_y \sin\delta
\end{aligned}
\tag{II.16}
$$

and, taking into account (II.6)

$$
\begin{aligned}
s_0 &= I \\
s_1 &= I\cos 2\psi \cos 2\chi = I\cos 2\alpha \\
s_2 &= I\cos 2\psi \sin 2\chi = I\sin 2\alpha \cos\delta \\
s_3 &= I\sin 2\psi = I\sin 2\alpha \sin\delta
\end{aligned}
\tag{II.17}
$$

It is important to point out that in this case of totally polarized light the following relation is satisfied

$$
s_0^2 = s_1^2 + s_2^2 + s_3^2 \tag{II.18}
$$

In general, light is presented as a superposition of a great number of simple wavelets with independent phases. The incoherent superposition of any number of light beams is characterized by a Stokes vector that is the sum of the Stokes vectors associated with them. The Stokes parameters of the total beam are

$$
s_0 = \sum_i s_0^i, \quad s_1 = \sum_i s_1^i, \quad s_2 = \sum_i s_2^i, \quad s_3 = \sum_i s_3^i, \tag{II.19}
$$

where the superscript "*i*" denotes each independent simple wave.

According to (II.19), the whole light beam is partially polarized, and its Stokes parameters can be obtained as follows [28]

$$
\begin{aligned}
s_0 &= \left\langle A_x^2 + A_y^2 \right\rangle \\
s_1 &= \left\langle A_x^2 - A_y^2 \right\rangle \\
s_2 &= \left\langle 2A_x A_y \cos\delta \right\rangle \\
s_3 &= \left\langle 2A_x A_y \sin\delta \right\rangle
\end{aligned}
\tag{II.20}
$$

where the brackets indicate the temporal average of each parameter.

The expressions (II.20) can be considered as the most general definitions for the Stokes parameters, which are subject to the condition

$$
s_0^2 \geq s_1^2 + s_2^2 + s_3^2 \tag{II.21}
$$

The equality is satisfied only for totally polarized light. In the case of natural light (unpolarized light), the averages are zero except for s_0, and the corresponding Stokes vector is

$$
\mathbf{S_N} = \begin{pmatrix} I_N \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{II.22}
$$

It is worth remembering now the optical equivalence principle of the states of polarization, which can be formulated as follows: "*By means of any physical experiment, is impossible to distinguish among several states of polarization of light that are incoherent sums of different pure states with the same Stokes vector associated*". [27]

According to this principle, a beam of partially polarized light can be considered as the incoherent superposition of two beams, one totally polarized, and the other one unpolarized. In the SMF formalism this fact is expressed as follows

$$
\mathbf{S} = \mathbf{S_P} + \mathbf{S_N} \tag{II.23}
$$

where

$$\mathbf{S} \equiv \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad \mathbf{S_P} \equiv \begin{pmatrix} I_P \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad \mathbf{S_N} \equiv \begin{pmatrix} I_N \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{II.24.a}$$

with

$$I_P = \left(s_1^2 + s_2^2 + s_3^2\right)^{1/2}, \quad I_N = s_0 - I_P \tag{II.4.b}$$

The degree of polarization, *G*, for a light beam with a Stokes vector **S** is defined as

$$G = \frac{I_P}{s_0} \tag{II.25}$$

We are also interested in defining the positive semidefinite quadratic form

$$F = s_0^2 - s_1^2 - s_2^2 - s_3^2 \tag{II.26}$$

which is related with *G* as follows

$$F = s_0^2\left(1 - G^2\right) \tag{II.27}$$

or

$$G = \left(1 - \frac{F}{s_0^2}\right)^{1/2} \tag{II.28}$$

The values of the quantities *G* and *F* are restricted by the following limits

$$0 \le G \le 1 \tag{II.29.a}$$

$$0 \le F \le s_0^2 \tag{II.29.b}$$

Thus, for totally polarized light, $G = 1,\ F = 0$; and for natural light $G = 0,\ F = s_0^2$.

A Stokes vector can be defined in terms of the total intensity *I*, the degree of polarization *G*, the azimuth χ and the ellipticity Ψ of the corresponding light beam

$$\mathbf{S} = I \begin{pmatrix} 1 \\ G\cos 2\psi \cos 2\chi \\ G\cos 2\psi \sin 2\chi \\ G\sin 2\psi \end{pmatrix} \tag{II.30}$$

This expression shows that the Stokes vector contains all the information about the polarization ellipse and the degree of polarization. However, the Stokes vector, unlike the Jones vector, does not contain information about the absolute phase of the corresponding light beam.

In the SMF formalism, linear optical systems are represented by means of 4x4 real matrices (Mueller matrices). These matrices are generically denoted as

$$\mathbf{M} \equiv \left(m_{ij}\right) \quad i,j = 0,1,2,3 \tag{II.31}$$

and contain 16 elements m_{ij}, generally independent.

When a light beam with a Stokes vector **S** passes through a medium that is characterized by the Mueller matrix **M**, the vector **S'** associated with the emerging beam is given by

$$\mathbf{S}' = \mathbf{MS} \tag{II.32}$$

As in JCF formalism, the Mueller matrix of a series of optical media is obtained as the product of the associated Mueller matrices [2].

On the other hand, the Mueller matrix associated with a group of optical media, which are passed through in parallel by an incoherent light beam, is given by the sum of the Mueller matrices associated with these media [27].

A Mueller matrix can represent any optical medium that affect to any parameter related with the Stokes vector associated with the incoming light beam. So, for example, all kind of retarders (linear, circular and elliptic), total or partial polarizers (linear, circular and elliptic), systems that depolarizes the light, or any complicated combination of them can be represented in SMF formalism. However, those media that introduce a uniform phase shift on light passing through them (phase plate) cannot be represented by means of the SMF formalism.

Hereafter we will use **S** and **M** to indicate Stokes vectors and Mueller matrices respectively. It must be understood as N-type Mueller matrices those corresponding to N-type optical media.

II.4 Coherency matrix and coherency vector formalisms

Let us consider a monochromatic light beam, characterized by an electric field vector **E** that, in general, can be thought as a superposition of vectors like (II.4), but with different phases δ_x, δ_y. We call coherency matrix ρ associated with such a light beam to the following [29]

$$\boldsymbol{\rho} = \langle \boldsymbol{\varepsilon} \times \boldsymbol{\varepsilon}' \rangle = \begin{pmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{pmatrix} = \begin{pmatrix} \langle A_x^2 \rangle & \langle A_x A_y e^{i\delta} \rangle \\ \langle A_x A_y e^{-i\delta} \rangle & \langle A_y^2 \rangle \end{pmatrix} \tag{II.33}$$

where the brackets indicate temporal average and × denotes the Kronecker product.

We denote the elements of ρ as follows

$$\boldsymbol{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_4 & \rho_3 \end{pmatrix} \tag{II.34}$$

The matrix ρ is a Hermitian matrix and it is defined on the basis of the same parameters as the Stokes vector **S**.

In fact, it is straightforward to prove the following relations between the elements of ρ and **S** associated with the same light beam [28,30].

$$\begin{aligned} s_0 &= \rho_1 + \rho_2 \\ s_1 &= \rho_1 - \rho_2 \\ s_2 &= \rho_3 + \rho_4 \\ s_3 &= i(\rho_3 - \rho_4) \end{aligned} \tag{II.35}$$

or

$$\begin{aligned} \rho_1 &= \tfrac{1}{2}(s_0 + s_1) \\ \rho_2 &= \tfrac{1}{2}(s_0 - s_1) \\ \rho_3 &= \tfrac{1}{2}(s_2 - is_3) \\ \rho_4 &= \tfrac{1}{2}(s_2 + is_3) \end{aligned} \tag{II.36}$$

The incoherent superposition of any number of light beams is characterized by a coherency matrix ρ, which is the sum of the coherency matrices ρ_i associated with the respective beams. Thus

$$\rho_1 = \sum_i \rho_1^i, \quad \rho_2 = \sum_i \rho_2^i, \quad \rho_3 = \sum_i \rho_3^i, \quad \rho_4 = \sum_i \rho_4^i, \tag{II.37}$$

where the superscript "*i*" denotes each independent simple wave.

The quadratic form F is now given by

$$F = 4 \det \boldsymbol{\rho} \tag{II.38}$$

and, according to (II.29.b)

$$0 \leq 4 \det \boldsymbol{\rho} \leq s_0^2 \tag{II.39}$$

When the light beam is totally polarized, then

$$\det \boldsymbol{\rho} = 0 \tag{II.40}$$

and when the light beam is totally unpolarized (natural light)

$$\det \boldsymbol{\rho} = \frac{1}{4} s_0^2 \tag{II.41}$$

Similarly to the treatment of the Stokes vector, any matrix $\boldsymbol{\rho}$ can be written as the sum of two coherency matrices as follows

$$\boldsymbol{\rho} = \boldsymbol{\rho}_{\mathbf{P}} + \boldsymbol{\rho}_{\mathbf{N}} \tag{II.42.a}$$

with

$$\det \boldsymbol{\rho}_{\mathbf{P}} = 0 \tag{II.2.b}$$

$$\det \boldsymbol{\rho}_{\mathbf{N}} = \frac{1}{4} I_N^2 \tag{II.42.c}$$

The matrix $\boldsymbol{\rho}_{\mathbf{N}}$ corresponds with a beam of unpolarized light with intensity I_N, and $\boldsymbol{\rho}_{\mathbf{P}}$ corresponds with a beam of totally polarized light.

II.4.1 N-type optical media

We call formalism of the coherency matrix, CMF, to that one that uses the coherency matrix to represent the state of polarization of light.

Let us consider a light beam with a coherency matrix $\boldsymbol{\rho} = \left\langle \boldsymbol{\varepsilon} \times \boldsymbol{\varepsilon}^{+} \right\rangle$ that passes through an N-type optical medium with a Jones matrix **J**. The emerging light beam will have an associated coherency matrix **ρ'** like this [27]

$$\boldsymbol{\rho}' = \left\langle \boldsymbol{\varepsilon} \times \boldsymbol{\varepsilon}'^{+} \right\rangle = \left\langle \mathbf{J}\boldsymbol{\varepsilon} \times \boldsymbol{\varepsilon}^{+}\mathbf{J}^{+} \right\rangle = \mathbf{J}\left\langle \boldsymbol{\varepsilon} \times \boldsymbol{\varepsilon}^{+} \right\rangle \mathbf{J}^{+} = \mathbf{J}\boldsymbol{\rho}\mathbf{J}^{+} \tag{II.43}$$

In the case of totally polarized light, the CMF formalism is equivalent to the JCF one, except for the fact that in the CMF it is no possible to handle information about the absolute phase of the wave of the light, but only about the characteristics of the polarization ellipse. In the present discussion we conclude that, when the phenomena are relative to totally polarized light, the JCF formalism is both simpler and more complete than the CMF one, given the fact that it contains information about the absolute phase.

The formalism JCF is not applicable to the study of phenomena with partially polarized light and N-type optical media, because this formalism does not allow the representation of states of partial polarization of the light. Thus, in general, the CMF formalism is more powerful than the JCF one, concerning the representation of states of light, but it is not concerning the representation of optical media because they are represented by Jones matrices in both formalisms.

II.4.2. G-type optical media

We call coherency vector, or density vector, **D** associated with a light beam, to the defined as follows

$$\mathbf{D} = \left\langle \mathbf{E} \times \mathbf{E}^{*} \right\rangle = \begin{pmatrix} \left\langle A_x^2 \right\rangle \\ \left\langle A_x A_y e^{-i\delta} \right\rangle \\ \left\langle A_x A_y e^{i\delta} \right\rangle \\ \left\langle A_y^2 \right\rangle \end{pmatrix} \tag{II.44}$$

We denote the elements of **D** as

$$\mathbf{D} \equiv \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{pmatrix} \tag{II.45}$$

They are given by the elements ρ_i of the coherency matrix associated with the same light beam

$$\begin{aligned} d_0 &= \rho_1 \\ d_1 &= \rho_3 \\ d_2 &= \rho_4 \\ d_3 &= \rho_2 \end{aligned} \tag{II.46}$$

Relations (II.35) and (II.36) can be expressed as the following vectorial form

$$\mathbf{S} = \mathbf{UD} \tag{II.47}$$

or

$$\mathbf{D} = \mathbf{U}^{-1}\mathbf{S} \tag{II.48}$$

where **U** is the following unitary matrix

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{pmatrix} \tag{II.49}$$

The vector **D** associated with a light beam that is an incoherent superposition of a certain number of light beams is given by the sum of the corresponding coherency vectors.

As we said above, the matrix $\boldsymbol{\rho}$, and therefore the vector **D**, contain exactly the same information than the corresponding Stokes vector **S**. Next we will see how the optical systems are characterized in the coherency vector formalisms (CVF), as a function of the corresponding Mueller matrices.

Let us consider an optical medium with the associated Mueller matrix **M**, with an incident light beam with Stokes vector **S** and coherency vector **D**. The Stokes, **S'**, and coherency, **D'**, vectors associated with the emerging beam fulfill the following relations

$$\mathbf{D}' = \mathbf{U}^{-1}\mathbf{S}' = \mathbf{U}^{-1}\mathbf{M}\mathbf{S} = \mathbf{U}^{-1}\mathbf{M}\mathbf{U}\mathbf{D} \qquad \text{(II.50)}$$

and consequently, for every Mueller matrix **M**, there is a unique matrix **V** in such a manner that

$$\mathbf{V} = \mathbf{U}^{-1}\mathbf{M}\mathbf{U} \qquad \text{(II.51)}$$

$$\mathbf{D}' = \mathbf{V}\mathbf{D} \qquad \text{(II.52)}$$

The values of the elements of the matrix ρ are restricted by the Hermiticity condition [30] $\boldsymbol{\rho} = \boldsymbol{\rho}^{+}$, i.e.

$$I_m(\rho_1) = I_m(\rho_2) = 0 \qquad \text{(II.53.a)}$$

$$\rho_3^* = \rho_{41} \qquad \text{(II.53.b)}$$

and, thus

$$I_m(d_0) = I_m(d_3) = 0 \qquad \text{(II.54.a)}$$

$$d_2^* = d_1 \qquad \text{(II.54.b)}$$

The components d_i' (i = 0,1,2,3) of the vector **D'** given by (II.52) are also restricted to the conditions (II.54). This implies that the 16 complex elements of a matrix **V** must satisfy a set of 16 restrictions in such a manner that, in general, it only depends on the 16 independent real parameters, in the same way as the Mueller matrix **M**.

By imposing the conditions (II.54) to the vectors **D** and **D'**, we see that the elements v_{ij} of the matrix **V** must satisfy the following restrictions [14]

$$\begin{aligned} \nu_{10} &= \nu_{20}^{*} \\ \nu_{01} &= \nu_{02}^{*} \\ \nu_{13} &= \nu_{23}^{*} \\ \nu_{31} &= \nu_{32}^{*} \\ \nu_{11} &= \nu_{12}^{*} \\ \nu_{21} &= \nu_{12}^{*} \end{aligned}$$
$$I_m(\nu_{00}) = I_m(\nu_{03}) = I_m(\nu_{30}) = I_m(\nu_{33}) = 0 \tag{II.55}$$

According to these expressions, a generic matrix **V** is characterized by 10 parameters corresponding to the real parts, and 6 parameters corresponding to the imaginary parts.

There is a total equivalence between the formalisms CVF and SMF. Both of them are equally powerful in any concrete case. Usually SMF is more practical because it only uses real numbers. The CVF formalism is especially useful to express the calculations or the results in relation to the coherency matrix.

Hereafter it must be understood that the use of the letters **D** and **V** corresponds to the density vectors and matrices of the CVF formalism. Likewise, we will say that a matrix V is N-type when it corresponds to an N-type optical medium.

II.5 Relations between the different formalisms

II.5.1 Some formal considerations

In the space of complex matrices 2x2 we can consider the base formed by the matrices [3, 18]

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{II.56}$$

The Pauli matrices $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3$ are usually grouped in the following matricial vector

$$\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3) \tag{II.57}$$

A coherency matrix $\boldsymbol{\rho}$ can be expressed as [27]

$$\boldsymbol{\rho} = \frac{1}{2}\sum_{i=0}^{3} s_i \boldsymbol{\sigma}_i \quad \text{(II.58)}$$

Taking into account the fact that the matrix $\boldsymbol{\rho}$ and the matrices $\boldsymbol{\sigma}_\mathbf{i}$ (i = 0, 1, 2, 3) are Hermitian, it is easy to prove that the coefficients s_i (i = 0, 1, 2, 3) must be real [27, 31].

If we compare the expressions (II.58) and (II.36) we see that the coefficients s_i are just the Stokes parameters corresponding to the matrix $\boldsymbol{\rho}$, and the following relation is fulfilled

$$s_i = \operatorname{tr}\left(\boldsymbol{\rho}\boldsymbol{\sigma}_i\right) \quad \text{(II.59)}$$

We can consider the expression (II.58) as the development of the density matrix $\boldsymbol{\rho}$ in a complete set of orthogonal observables ($\boldsymbol{\sigma}_i$) in such a manner that the coefficients s_i correspond, except for a constant, with the eigenvalues of these operators.

Let us consider a light beam that has associated a density matrix $\boldsymbol{\rho}$ and a Stokes vector **S**, and that passes through an N-type optical medium characterized by a Jones matrix **J**. Then, the Stokes vector **S'** associated with the emerging light beam is given by

$$s_i' = \operatorname{tr}\left(\boldsymbol{\sigma}_i\boldsymbol{\rho}'\right) = \operatorname{tr}\left(\boldsymbol{\sigma}_i\mathbf{J}\boldsymbol{\rho}\mathbf{J}^+\right) = \frac{1}{2}\operatorname{tr}\left(\boldsymbol{\sigma}_i\mathbf{J}\sum_{j=0}^{3} s_j\boldsymbol{\sigma}_j\mathbf{J}^+\right) = \frac{1}{2}\operatorname{tr}\sum_{j=0}^{3}\left(\boldsymbol{\sigma}_i\mathbf{J}\boldsymbol{\sigma}_j\mathbf{J}^+\right)s_j = \sum_{j=0}^{3} m_{ij}s_j \quad \text{(II.60)}$$

where the matrix **M**, whose elements are

$$m_{ij} = \frac{1}{2}\operatorname{tr}\sum_{j=0}^{3}\left(\boldsymbol{\sigma}_i\mathbf{J}\boldsymbol{\sigma}_j\mathbf{J}^+\right) \quad \text{(II.61)}$$

is just the Mueller matrix associated with the same optical medium represented by the Jones matrix **J**.

II.5.2 Relations about the characterization of light

In the case of totally polarized light it is easy to prove the following relation [3]

$$s_j = \boldsymbol{\varepsilon}^+\boldsymbol{\sigma}_j\boldsymbol{\varepsilon} \quad \text{(II.62)}$$

or, in an explicit way

$$\begin{aligned} s_0 &= |\varepsilon_1|^2 + |\varepsilon_2|^2 \\ s_1 &= |\varepsilon_1|^2 - |\varepsilon_2|^2 \\ s_2 &= 2\varepsilon_1\varepsilon_2 \cos\delta \\ s_3 &= 2\varepsilon_1\varepsilon_2 \sin\delta \end{aligned} \tag{II.63.a}$$

with

$$\delta = \left(\arg\varepsilon_2 - \arg\varepsilon_1\right) \tag{II.63.b}$$

Reciprocally

$$|\varepsilon_1|^2 = \frac{1}{2}\left(s_0 + s_1\right), \quad |\varepsilon_1|^2 = \frac{1}{2}\left(s_0 - s_1\right), \quad \tan\delta = \frac{s_3}{s_2} \tag{II.64}$$

The relations between the Jones vector, the coherency matrix and the coherency vector associated with the same beam of totally polarized light are the following

$$\begin{aligned} \rho_1 &= d_0 = |\varepsilon_1|^2 \\ \rho_2 &= d_3 = |\varepsilon_2|^2 \\ \rho_3 &= d_1 = \varepsilon_1\varepsilon_2 e^{-i\delta} \\ \rho_4 &= d_2 = \varepsilon_1\varepsilon_2 e^{i\delta} \end{aligned} \tag{II.65}$$

or, reciprocally

$$\begin{aligned} &|\varepsilon_1|^2 = \rho_1 = d_0 \\ &|\varepsilon_2|^2 = \rho_2 = d_3 \\ &\delta = \arg\rho_4 = -\arg\rho_3 = \arg d_2 = -\arg d_1 \end{aligned} \tag{II. 66}$$

The relations between the coherency matrix and the Stokes parameters were considered in (II.35) and (II.36), and are valid regardless the value of the degree of polarization of the light beam.

II.5.3 Relations about the characterization of optical media

An N-type optical medium has associated a Jones matrix **J** and a Mueller matrix **M**. Let us consider the Jones vector **ε** and the Stokes vector **S** associated with the incident light beam over the medium.

The emerging beam is also characterized by the Jones and Stokes vectors **ε'** and **S'** respectively. These vectors are obtained, according to (II.12) and (II.32), as follows

$$\boldsymbol{\varepsilon}'_{\mathbf{k}} = \sum_{l=1}^{2} J_{kl}\boldsymbol{\varepsilon}_l, \quad k = 1,2 \tag{II. 67}$$

$$\mathbf{s}'_{\mathbf{i}} = \sum_{j=0}^{3} m_{ij}\mathbf{s}_j, \quad i = 0,1,2,3 \tag{II. 68}$$

Taking into account (II.58), we can write (II.68) as follows

$$\boldsymbol{\varepsilon}'^{+}\boldsymbol{\sigma}_i\boldsymbol{\varepsilon}' = \sum_{j=0}^{3} m_{ij}\left(\boldsymbol{\varepsilon}^{+}\boldsymbol{\sigma}_j\boldsymbol{\varepsilon}\right) \tag{II. 69}$$

or

$$\boldsymbol{\varepsilon}'^{+}\boldsymbol{\sigma}_i\boldsymbol{\varepsilon}' == \boldsymbol{\varepsilon}^{+}\left(\sum_{j=0}^{3} m_{ij}\boldsymbol{\sigma}_j\right)\boldsymbol{\varepsilon} \tag{II. 70}$$

From (II.69) and (II.12) we obtain

$$\boldsymbol{\varepsilon}'^{+}\boldsymbol{\sigma}_i\boldsymbol{\varepsilon}' == \boldsymbol{\varepsilon}^{+}\left(\mathbf{J}\boldsymbol{\sigma}_j\mathbf{J}^{+}\right)\boldsymbol{\varepsilon} \tag{II. 71}$$

Together with (II.70), this expression leads to

$$\mathbf{J}\boldsymbol{\sigma}_j\mathbf{J}^{+} = \sum_{j=0}^{3} m_{ij}\boldsymbol{\sigma}_j, \quad i = 0,1,2,3 \tag{II. 72}$$

This last expression is useful to obtain the elements of one matrix as a function of the other in the following manner

$$
\begin{aligned}
2m_{00} &= J_{11}^{*}J_{11} + J_{12}^{*}J_{12} + J_{21}^{*}J_{21} + J_{22}^{*}J_{22} \\
2m_{01} &= J_{11}^{*}J_{11} + J_{21}^{*}J_{21} - J_{12}^{*}J_{12} - J_{22}^{*}J_{22} \\
2m_{02} &= J_{11}^{*}J_{12} + J_{21}^{*}J_{22} + J_{12}^{*}J_{11} + J_{22}^{*}J_{21} \\
2m_{03} &= i\left(J_{11}^{*}J_{12} + J_{21}^{*}J_{22} - J_{12}^{*}J_{11} - J_{22}^{*}J_{21}\right) \\
2m_{10} &= J_{11}^{*}J_{11} + J_{12}^{*}J_{12} - J_{21}^{*}J_{21} - J_{22}^{*}J_{22} \\
2m_{11} &= J_{11}^{*}J_{11} + J_{22}^{*}J_{22} - J_{21}^{*}J_{21} - J_{12}^{*}J_{12} \\
2m_{12} &= J_{12}^{*}J_{11} + J_{11}^{*}J_{12} - J_{22}^{*}J_{21} - J_{21}^{*}J_{22} \\
2m_{13} &= i\left(J_{11}^{*}J_{12} + J_{22}^{*}J_{21} - J_{21}^{*}J_{22} - J_{12}^{*}J_{11}\right) \\
2m_{20} &= J_{11}^{*}J_{21} + J_{21}^{*}J_{11} + J_{12}^{*}J_{22} + J_{22}^{*}J_{12} \\
2m_{21} &= J_{11}^{*}J_{21} + J_{21}^{*}J_{11} - J_{12}^{*}J_{22} - J_{22}^{*}J_{12} \\
2m_{22} &= J_{11}^{*}J_{22} + J_{21}^{*}J_{12} + J_{12}^{*}J_{21} + J_{22}^{*}J_{11} \\
2m_{23} &= i\left(J_{11}^{*}J_{22} + J_{21}^{*}J_{12} - J_{12}^{*}J_{21} - J_{22}^{*}J_{11}\right) \\
2m_{30} &= i\left(J_{21}^{*}J_{11} + J_{22}^{*}J_{12} - J_{11}^{*}J_{21} - J_{12}^{*}J_{22}\right) \\
2m_{31} &= i\left(J_{21}^{*}J_{11} + J_{12}^{*}J_{22} - J_{11}^{*}J_{21} - J_{22}^{*}J_{12}\right) \\
2m_{32} &= i\left(J_{21}^{*}J_{12} + J_{22}^{*}J_{11} - J_{11}^{*}J_{22} - J_{12}^{*}J_{21}\right) \\
2m_{33} &= J_{22}^{*}J_{11} + J_{11}^{*}J_{22} - J_{12}^{*}J_{21} - J_{21}^{*}J_{12}
\end{aligned}
\tag{II. 73}
$$

and reciprocally, by denoting the elements J_{kl} ($k,l = 1, 2$) in polar form as

$$J_{kl} = \left|J_{kl}\right| e^{i\theta_{kl}} \tag{II. 74.a}$$

it can be proved the following relations

$$
\begin{aligned}
2\left|J_{11}\right|^2 &= m_{00} + m_{01} + m_{10} + m_{11} \\
2\left|J_{12}\right|^2 &= m_{00} - m_{01} + m_{10} - m_{11} \\
2\left|J_{21}\right|^2 &= m_{00} + m_{01} - m_{10} - m_{11} \\
2\left|J_{22}\right|^2 &= m_{00} - m_{01} - m_{10} + m_{11}
\end{aligned}
\tag{II. 74.b}
$$

$$\cos\left(\theta_{12}-\theta_{11}\right)=\frac{m_{02}+m_{12}}{\left[\left(m_{00}+m_{10}\right)^2-\left(m_{01}+m_{11}\right)^2\right]^{1/2}}$$

$$\sin\left(\theta_{12}-\theta_{11}\right)=\frac{-\left(m_{03}+m_{13}\right)}{\left[\left(m_{00}+m_{10}\right)^2-\left(m_{01}+m_{11}\right)^2\right]^{1/2}}$$

$$\cos\left(\theta_{21}-\theta_{11}\right)=\frac{m_{20}+m_{21}}{\left[\left(m_{00}+m_{01}\right)^2-\left(m_{10}+m_{11}\right)^2\right]^{1/2}}$$

$$\sin\left(\theta_{21}-\theta_{11}\right)=\frac{m_{30}+m_{31}}{\left[\left(m_{00}+m_{01}\right)^2-\left(m_{10}+m_{11}\right)^2\right]^{1/2}}$$

$$\cos\left(\theta_{22}-\theta_{11}\right)=\frac{m_{22}+m_{33}}{\left[\left(m_{00}+m_{11}\right)^2-\left(m_{10}+m_{11}\right)^2\right]^{1/2}}$$

$$\sin\left(\theta_{22}-\theta_{11}\right)=\frac{m_{32}+m_{23}}{\left[\left(m_{00}+m_{11}\right)^2-\left(m_{10}+m_{01}\right)^2\right]^{1/2}}$$

It should be noted that the transformation of the Jones matrix in the Mueller matrix provokes the loss of information concerning the global retardation introduced by the corresponding optical system.

A more compacted way to present the relations (II.73) is the following [4, 9, 32]

$$\mathbf{M}=\begin{pmatrix} \tfrac{1}{2}\left(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2\right) & \tfrac{1}{2}\left(\alpha_1^2-\alpha_2^2-\alpha_3^2+\alpha_4^2\right) & \beta_{13}+\beta_{42} & -\gamma_{13}-\gamma_{42} \\ \tfrac{1}{2}\left(\alpha_1^2-\alpha_2^2+\alpha_3^2-\alpha_4^2\right) & \tfrac{1}{2}\left(\alpha_1^2+\alpha_2^2-\alpha_3^2-\alpha_4^2\right) & \beta_{13}-\beta_{42} & -\gamma_{13}+\gamma_{42} \\ \beta_{14}+\beta_{32} & \beta_{14}-\beta_{32} & \beta_{12}+\beta_{34} & -\gamma_{12}+\gamma_{34} \\ \gamma_{14}+\gamma_{32} & \gamma_{14}-\gamma_{32} & \gamma_{12}+\gamma_{34} & \beta_{12}-\beta_{34} \end{pmatrix} \quad \text{(II. 75.a)}$$

where

$$\begin{aligned} &\alpha_i^2=J_iJ_i^*=\left|J_i\right|^2, \quad i=1,2,3,4 \\ &\beta_{ij}=\beta_{ji}=\mathrm{Re}\left(J_iJ_j^*\right)=\mathrm{Re}\left(J_jJ_i^*\right) \\ &\gamma_{ij}=-\gamma_{ji}=\mathrm{Im}\left(J_iJ_j^*\right)=\mathrm{Im}\left(J_jJ_i^*\right), \quad i,j=1,2,3,4 \end{aligned} \quad \text{(II. 75.b)}$$

The matrix (II.75) can be obtained directly from (II.61).

If a Mueller matrix **M** corresponds to a Jones matrix **J**, the former has the form (II.75), and it is easy to prove that the Mueller matrices $\mathbf{M}^T$ and **M'** correspond to the Jones matrices $\mathbf{J}^+$ and $\mathbf{J}^T$, where **M'** is given by

$$\mathbf{M}' = \begin{pmatrix} m_{00} & m_{10} & m_{20} & -m_{30} \\ m_{01} & m_{11} & m_{21} & -m_{31} \\ m_{02} & m_{12} & m_{22} & -m_{32} \\ -m_{03} & -m_{13} & -m_{23} & m_{33} \end{pmatrix} \quad \text{(II. 76)}$$

which can be written as

$$\mathbf{M}' = \mathbf{Q}\mathbf{M}^T\mathbf{Q} \quad \text{(II. 77.a)}$$

with

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{(II. 77.b)}$$

The diagonal matrix Q is orthogonal, with $\det \mathbf{Q} = -1$, and does not correspond to any optical system with real physical entity.

Next we will search the relations between the Jones matrix **J** and the matrix **V** associated with the same N-type optical medium.

If we take into account (II.12) and (II.44), we see that [28]

$$\mathbf{V} = \mathbf{J} \times \mathbf{J}^* . \quad \text{(II. 78)}$$

That is to say

$$\mathbf{V} = \begin{pmatrix} J_1 J_1^* & J_1 J_3^* & J_3 J_1^* & J_3 J_3^* \\ J_1 J_4^* & J_1 J_2^* & J_3 J_4^* & J_3 J_2^* \\ J_4 J_1^* & J_4 J_3^* & J_2 J_1^* & J_2 J_3^* \\ J_4 J_4^* & J_4 J_2^* & J_2 J_4^* & J_2 J_2^* \end{pmatrix} . \quad \text{(II. 79)}$$

Reciprocally, we obtain the elements $J_i = |J_i| e^{i\theta_i}$ as a function of the elements v_{ij}

$$
\begin{aligned}
|J_1|^2 &= \nu_{00} \\
|J_2|^2 &= \nu_{33} \\
|J_3|^2 &= \nu_{03} \\
|J_4|^2 &= \nu_{30} \\
\theta_1 - \theta_2 &= \arg(\nu_{11}) = -\arg(\nu_{22}) \\
\theta_1 - \theta_3 &= \arg(\nu_{01}) = -\arg(\nu_{02}) \\
\theta_1 - \theta_4 &= \arg(\nu_{10}) = -\arg(\nu_{20})
\end{aligned}
\qquad \text{(II. 80)}
$$

The relations (II.75) and (II.79) are only valid for N-type optical media, because otherwise the Jones matrices are not defined.

Finally, we will see the relations between the matrices **M** and **V** that correspond to the same G-type optical medium.

According to (II.51), we know that $\mathbf{V} = \mathbf{U}^{-1}\mathbf{M}\mathbf{U}$, where **U** is the matrix given in (II.49). As **U** is a unitary matrix we can write

$$\mathbf{M} = \mathbf{U}\mathbf{V}\mathbf{U}^{-1} \qquad \text{(II. 81)}$$

Developing (II.51) and (II.81) in the explicit form we obtain [41]

$$
\mathbf{V} = \frac{1}{2}\begin{pmatrix}
m_{00}+m_{01}+m_{10}+m_{11} & m_{02}+m_{12}+i\left(m_{03}+m_{13}\right) & m_{02}+m_{12}-i\left(m_{03}+m_{13}\right) & m_{00}-m_{01}+m_{10}+m_{11} \\
m_{20}+m_{21}-i\left(m_{30}+m_{31}\right) & m_{22}+m_{33}+i\left(m_{23}+m_{32}\right) & m_{22}-m_{33}-i\left(m_{23}+m_{32}\right) & m_{20}-m_{21}-i\left(m_{30}-m_{31}\right) \\
m_{20}+m_{21}+i\left(m_{30}+m_{31}\right) & m_{22}-m_{33}+i\left(m_{23}+m_{32}\right) & m_{22}+m_{33}-i\left(m_{23}-m_{32}\right) & m_{20}-m_{21}+i\left(m_{30}-m_{31}\right) \\
m_{00}+m_{01}-m_{10}-m_{11} & m_{02}-m_{12}+i\left(m_{03}-m_{13}\right) & m_{02}-m_{12}-i\left(m_{03}-m_{13}\right) & m_{00}-m_{01}-m_{10}+m_{11}
\end{pmatrix}
\qquad \text{(II. 82.a)}
$$

and, reciprocally

$$
\mathbf{M} = \frac{1}{2}\begin{pmatrix}
\nu_{00}+\nu_{03}+\nu_{30}+\nu_{33} & \nu_{00}+\nu_{03}+\nu_{30}-\nu_{33} & \nu_{01}+\nu_{02}+\nu_{31}+\nu_{32} & -i\left(\nu_{01}-\nu_{02}+\nu_{31}-\nu_{32}\right) \\
\nu_{00}+\nu_{03}-\nu_{30}-\nu_{33} & \nu_{00}-\nu_{03}-\nu_{30}+\nu_{33} & \nu_{01}+\nu_{02}-\nu_{31}-\nu_{32} & -i\left(\nu_{01}-\nu_{02}-\nu_{31}+\nu_{32}\right) \\
\nu_{10}+\nu_{13}+\nu_{20}+\nu_{23} & \nu_{10}-\nu_{13}+\nu_{20}-\nu_{23} & \nu_{11}+\nu_{12}+\nu_{21}+\nu_{22} & -i\left(\nu_{11}-\nu_{12}+\nu_{21}-\nu_{22}\right) \\
-i\left(\nu_{10}-\nu_{20}+\nu_{13}-\nu_{23}\right) & -i\left(\nu_{10}-\nu_{20}-\nu_{13}+\nu_{23}\right) & -i\left(\nu_{11}+\nu_{12}-\nu_{21}+\nu_{22}\right) & \nu_{11}-\nu_{12}-\nu_{21}+\nu_{22}
\end{pmatrix}
\qquad \text{(II. 82.b)}
$$

From (II.79) and (II.81) we see that if the Jones and Mueller matrices **J** and **M** correspond to a matrix **V**, then the matrices $\mathbf{J}^{+}$ and $\mathbf{M}^{T}$ correspond to $\mathbf{V}^{+}$, and $\mathbf{J}^{T}$, **M'** correspond to $\mathbf{V}^{T}$.

The figures (II.3) and (II.4) show schematically the relations between the different formalisms.

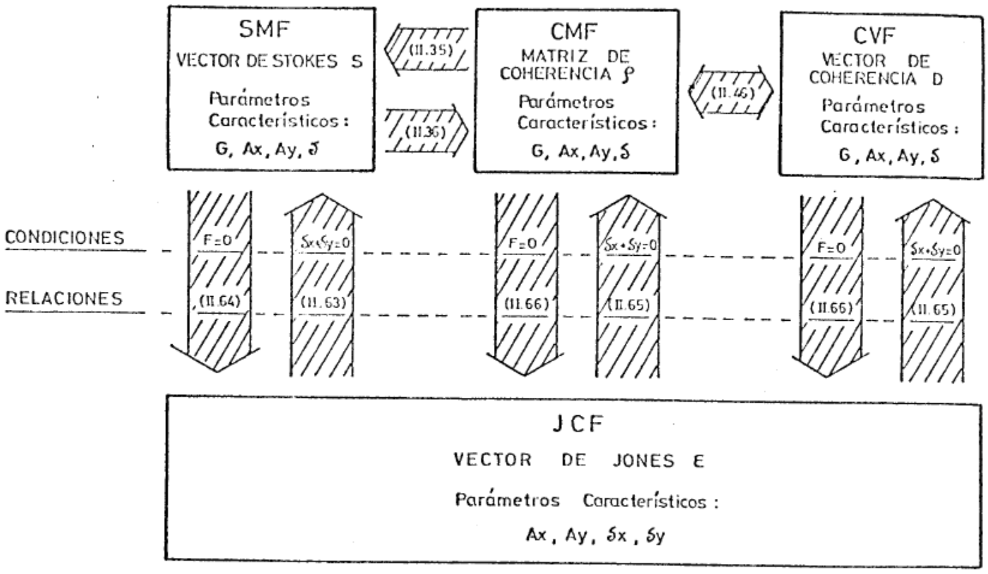


Fig. II.3: Scheme of the relations, regarding the characterization of the light, among the formalisms SMF, CMF, CVF and JCF.

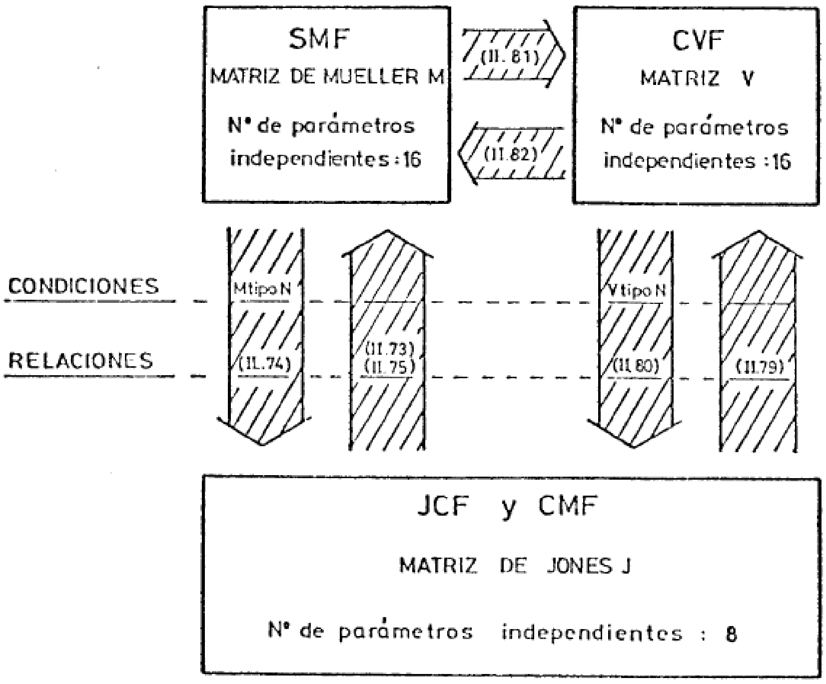


Fig. II.4: Scheme of the relations, regarding the characterization of optical media, among the formalisms SMF, CMF, CVF and JCF.

II.6. Optical media. Notation

Along our work, we will study systems composed of N-type optical media. The set of these media can be divided into two categories, regarding the nature of the effects produced over the polarized light. Some of them produce retardation between two orthogonal states of polarization that are invariant under the action of the considered medium (retarders), and others produce a selective absorption or reflection (polarizers).

Both of them, retarders and polarizers, total or partial ones, can be linear, circular or elliptic, depending on the invariant eigenstates of polarization.

As we will see, any elliptic retarder is equivalent to a system composed of a linear retarder and a rotator (circular retarder). This fact allows us to express every phenomenon of retardation by means of linear retarders and rotators.

Otherwise, we will see that a partial (total) polarizer, circular or elliptic, is optically equivalent to a certain combination of two linear retarders and a linear partial (total) polarizer.

These considerations let us to state that any optical system composed of N-type elements, is optically equivalent to a certain combination of linear retarders, rotators and linear partial polarizers.

We denote by L(θ,δ) the linear retarders with a phase retardation δ and an angle θ of its fast axis respect to a prefixed reference axis X. By R(γ), we understand a circular retarder that introduces a retardation of 2γ between its two eigenstates of circular polarization. Finally, a linear partial polarizer with principal coefficients of transmission in amplitude p_1, p_2, and angle α of its axis of polarization with the reference axis X, will be denoted by P(α, p_1, p_2). If the polarizer is total, that is, $p_2 = 0$, we will denote it by P (α).

The matrices associated with linear retarders, linear partial polarizers and linear total polarizers will be denoted by $\mathbf{B_L}(\theta, \delta)$, $\mathbf{B_R}(\gamma)$, $\mathbf{B_P}(\alpha, p_1, p_2)$ and $\mathbf{B_P}(\alpha)$, respectively, where **B** can be a Mueller matrix **M**, a Jones matrix **J** or a matrix **V** depending on the formalism considered.

II.6.1. Partial polarizer

In JCF formalism, a partial polarizer is characterized by a Hermitian matrix with no negative real eigenvalues [1]. These eigenvalues are just the principal coefficients of the transmission in amplitude p_1, p_2, of the polarizer. A partial polarizer is called linear, circular or elliptic depending on the eigenvectors of the associated Jones matrix **H**, which can correspond with linear, circular or elliptical polarizations [3]. We exclude for the following discussion the case of total polarizers (seen on the next section), so we will consider that p_1, $p_2 \neq 0$. We will also suppose, for the sake of concreteness, that $p_1 > p_2$.

The coefficients p_1, p_2 can take values in the following ranges

$$0 < p_2 < p_1 \leq 1 \tag{II. 83}$$

-There is an erratum in the original that has been corrected here-

If we take into account (II.83) and the fact that det **H** = $p_1 p_2$, we see that

$$0 < \det \mathbf{H} < 1 \tag{II. 84}$$

This means that the matrix **H** has an inverse $\mathbf{H}^{-1}$. However, $\mathbf{H}^{-1}$ is not a Jones matrix because

$$\det \mathbf{H}^{-1} = \frac{1}{\det \mathbf{H}} > 1 \tag{II. 85}$$

The interpretation of this fact is clear, because the passing of the light through a polarizer produces a loss of intensity in the emerging beam, which cannot be compensated by any passive optical medium. However there is a physically realizable optical medium whose Jones matrix is

$$\mathbf{H}' = \lambda \mathbf{H}^{-1} \tag{II. 86}$$

where λ is the real number such as $\lambda < \det \mathbf{H}$, and thus

$$\mathbf{H}\mathbf{H}' = \lambda \mathbf{I} \tag{II. 87}$$

The partial polarizer of the Jones matrix **H'** can be considered as the inverse of the Jones matrix **H**, in the sense that a light beam passing through them successively presents at the exit the same state of polarization as at the input, although a loss in the intensity of the light beam is produced.

In the SMF formalism, any partial polarizer is represented by a Mueller matrix **K** that is symmetric with four eigenvalues k_1, k_2, $(k_1k_2)^{1/2}$ (double). The eigenvalue $(k_1k_2)^{1/2}$ corresponds to eigenvectors **S**, **S'**, with $s_0 = s'_0 = 0$, so that these eigenvectors have not physical meaning [29]. The other two eigenvalues k_1, k_2, correspond to the principal coefficients of transmission in intensity $k_1 = p_1^2$ and $k_2 = p_2^2$.

The matrix **K** is such that $\det \mathbf{K} = k_1^2 k_2^2$, and, as in the case of the matrix **H**, the following condition is fulfilled

$$0 < \det \mathbf{K} < 1 \qquad \text{(II. 88)}$$

It is worth mentioning that if **H** and **K** correspond to the same partial polarizer, then

$$\det \mathbf{K} = \left(\det \mathbf{H}\right)^4 \qquad \text{(II. 89)}$$

From (II.88) is deduced that there is a matrix $\mathbf{K}^{-1}$ that does not represent any passive optical medium. However, the matrix $\mathbf{K}' = \mu \mathbf{K}^{-1}$, with μ < det**K**, does represent a passive optical medium that, regarding the polarization, produces an inverse optical effect to the produced by the polarizer corresponding with the matrix **K**.

In some occasions, for the sake of systematic and formal treatment of the matrices associated with partial polarizers, is interesting to normalize them by dividing by their determinants, so that they have the unity as the determinant. Once normalized, the matrices are denoted as follows

$$\mathbf{H_N} = \frac{1}{\det \mathbf{H}} \mathbf{H} \qquad \text{(II. 90.a)}$$

$$\mathbf{K_N} = \frac{1}{\det \mathbf{K}} \mathbf{K} \qquad \text{(II. 90.b)}$$

In JCF formalism, a linear polarizer is represented by a Jones matrix $\mathbf{H_P}$, which is diagonal as follows

$$\mathbf{H_P} = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \qquad \text{(II. 91)}$$

In SMF formalism, a linear polarizer is represented by the Mueller matrix $\mathbf{K_P}$, which transformed to be referred to their own axes, is [10,26]

$$\mathbf{K_P} = \frac{1}{2} \begin{pmatrix} p_1^2 + p_2^2 & p_1^2 - p_2^2 & 0 & 0 \\ p_1^2 - p_2^2 & p_1^2 + p_2^2 & 0 & 0 \\ 0 & 0 & 2p_1 p_2 & 0 \\ 0 & 0 & 0 & 2p_1 p_2 \end{pmatrix} \qquad \text{(II. 92)}$$

The matrix $\mathbf{K_P}$ can be written in diagonal form by means of the matrix **C** (called modal matrix) as follows

$$\mathbf{K_D} = \mathbf{C}\mathbf{K_P}\mathbf{C}^{-1} \tag{II. 93}$$

or

$$\mathbf{K_P} = \mathbf{C}^{-1}\mathbf{K_D}\mathbf{C} \tag{II. 94}$$

where

$$\mathbf{K_D} = \begin{pmatrix} p_1^2 & 0 & 0 & 0 \\ 0 & p_2^2 & 0 & 0 \\ 0 & 0 & p_1 p_2 & 0 \\ 0 & 0 & 0 & p_1 p_2 \end{pmatrix} \tag{II. 95}$$

and

$$\mathbf{C} = \mathbf{C}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \tag{II. 96}$$

According to the expression (II.96), the matrix $\mathbf{C}$ is orthogonal, because $\mathbf{C}\mathbf{C}^T = \mathbf{C}\mathbf{C} = \mathbf{C}\mathbf{C}^{-1} = \mathbf{I}$, and moreover we see that $\det \mathbf{C} = -1$.

The transformation (II.93) conserves the trace and, thus

$$\mathrm{tr}\mathbf{K_D} = \mathrm{tr}\mathbf{K_p} = \left(p_1 + p_2\right)^2 \tag{II. 97}$$

II.6.2. Total polarizers

The matrices $\mathbf{H_T}$ and $\mathbf{K_T}$, associated with a total polarizer (linear, circular or elliptic) in the formalisms JCF and SMF respectively, are characterized by having one zero eigenvalue and, consequently, they are singular matrices. Because of this fact, $\mathbf{H_T}$ and $\mathbf{K_T}$ cannot be normalized in the sense given in (II.90).

An interesting property of $\mathbf{H_T}$ and $\mathbf{K_T}$ is that they are idempotent ($\mathbf{H}_\mathbf{T}^2 = \mathbf{H}_\mathbf{T}$, $\mathbf{K}_\mathbf{T}^2 = \mathbf{K}_\mathbf{T}$). These matrices play the role of projectors in the of Jones and Stokes spaces respectively.

A linear total polarizer is represented by a Jones matrix $\mathbf{H_{TP}}$, which referring to their own polarization axes, is expressed as [1]

$$\mathbf{H_{TP}} = \begin{pmatrix} p_1 & 0 \\ 0 & 0 \end{pmatrix} \tag{II. 98}$$

and by the following Mueller matrix $\mathbf{K_{TP}}$ (also referring to their own axes)

$$\mathbf{K_{TP}} = \frac{p_1^2}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{II. 99}$$

which can be written as [10]

$$\mathbf{K_{TP}} = \mathbf{C}\mathbf{K_{TD}}\mathbf{C}^{-1} \tag{II. 100.a}$$

where

$$\mathbf{K_{TD}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{II. 100.b}$$

II.6.3. Ideal retarders

The Mueller matrix **R** associated with an ideal retarder (linear, circular or elliptic) has the property of leaving invariant the parameter s_0 (intensity), and produces a rotation of the vector (s_1, s_2, s_3) in the Poincaré sphere. This fact let us write **R** in the form

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & (\mathbf{\Omega}_{ij}) & \\ 0 & & & \end{pmatrix} \qquad \text{(II. 101)}$$

where the submatrix $\mathbf{\Omega} \equiv (\mathbf{\Omega}_{ij})$ is a 3x3 matrix associated with a generic rotation in the subspace that contains the variables s_1, s_2, s_3.

The set of 3x3 orthogonal matrices **Ω** with $\det \mathbf{\Omega} = +1$ constitutes a group dependent of three parameters called O_3^+ (group of rotations in the ordinary space). The three independent parameters in **Ω** can be, for example, the three Euler angles [34]. However, the azimuth χ and the ellipticity ψ of the two orthogonal eigenstates of polarization, which are invariant under the action of the retarder, with the retardation δ introduced between them, are more useful as parameters [35].

An ideal retarder is represented in the JCF formalism by a unitary matrix **U** such as $\det \mathbf{U} = +1$. This matrix **U** corresponds to a rotation of a certain angle ϕ of the Stokes vector in the Poincaré sphere around a certain axis whose direction is given by a unitary vector **û**. This let us write [36]

$$\mathbf{U} = \exp\left[\left(-i\frac{\phi}{2}\right)\hat{\mathbf{u}}\sigma\right] \qquad \text{(II. 102)}$$

The set of 2x2 unitary complex matrices **U** with $\det \mathbf{U} = +1$ constitutes a group called SU (2C) (special unitary group of 2x2 complex matrices) [31]. There is a biunivocal correspondence between the set made by pairs of matrices (**U**, -**U**) belonging to the group SU(2C), and the set of matrices **R** such as **Ω** belongs to the group O_3^+.

II.6.4. Non-ideal retarders

It is well known that, by the effect of multiple internal reflections, every linear retarder presents different transmittance for the two linear polarized eigenstates [37]. The effect is equivalent to that produced by an ideal linear retarder followed (or preceded) by a linear partial polarizer whose polarization axes are aligned with the ones of the retarder . The Mueller matrix $\mathbf{M_L}$ associated with a non-ideal linear retarder, referring to its own polarization axes, can be written as follows

$$\mathbf{M_L} = \mathbf{M_L}(0,\delta)\mathbf{M_P}(0,p_1,p_2) = \mathbf{M_P}(0,p_1,p_2)\mathbf{M_L}(0,\delta) \qquad \text{(II. 103)}$$

where δ is the characteristic effective phase retardation of the retarder, and p_1, p_2 are the principal coefficients of transmission in amplitude depending on the neutral lines (polarization axes) of the retarder.

The matrix $\mathbf{M_L}$ obtained by means of (II.103) is

$$\mathbf{M_L}(0,\delta,k_a,k_a') = \frac{1}{2k_a}\begin{pmatrix} 1+k & 1-k & 0 & 0 \\ 1-k & 1+k & 0 & 0 \\ 0 & 0 & 2\sqrt{k}\cos\delta & 2\sqrt{k}\sin\delta \\ 0 & 0 & -2\sqrt{k}\sin\delta & 2\sqrt{k}\cos\delta \end{pmatrix} \qquad \text{(II. 104.a)}$$

where

$$k_a \equiv p_1^2, \quad k_a' \equiv p_2^2, \quad k = \frac{k_a'}{k_a} \qquad \text{(II. 104.b)}$$

If the fast axis of the retarder presents an angle β with respect to the X axis of reference, the matrix associated with the retarder is given by

$$\mathbf{M_L}(p,\delta,k_a,k_a') = \mathbf{M_R}(-\beta)\mathbf{M_L}(p,\delta,k_a,k')\mathbf{M_R}(\beta) =$$

$$= \frac{1}{k_a}\begin{pmatrix} 1+k & (1-k)\cos 2\beta & (1-k)\sin 2\beta & 0 \\ (1-k)\cos 2\beta & (1+k)\cos^2 2\beta + 2\sqrt{k}\cos\delta\sin^2 2\beta & \left(1+k-2\sqrt{k}\cos\delta\right)\frac{1}{2}\sin 4\beta & -2\sqrt{k}\sin\delta\sin 2\beta \\ (1-k)\sin 2\beta & \left(1+k-2\sqrt{k}\cos\delta\right)\frac{1}{2}\sin 4\beta & (1+k)\sin^2 2\beta + 2\sqrt{k}\cos\delta\cos^2 2\beta & 2\sqrt{k}\sin\delta\cos 2\beta \\ 0 & 2\sqrt{k}\sin\delta\sin 2\beta & -2\sqrt{k}\sin\delta\cos 2\beta & 2\sqrt{k}\cos\delta \end{pmatrix}$$

(II. 105)

Hereafter we will use the following notation

$$\mathbf{M_L}(\beta,\delta,k) \equiv 2k_a\mathbf{M_L}(\beta,\delta,k_a,k_a') \qquad \text{(II.106)}$$

II.6.5. SL(2C) group and Lorentz group

The set of 2x2 complex matrices $\mathbf{A}$ with $\det \mathbf{A} = +1$ constitutes a group called SL(2C) (unimodular group of complex matrices 2x2). Every Jones matrix that is non-singular can be normalized in the way indicated in (II.90.a), and thus, once normalized it belongs to the SL(2C) group.

The set of 4x4 real matrices $\mathbf{L}$ leaving invariant the quadratic form F, makes a group called the Lorentz group. When it is also satisfied $\det \mathbf{L} = 1$ and $L_{00} \geq 1$, then it is called Lorentz ortochronous subgroup or restricted Lorentz subgroup "L_+". Every non-singular N-type Mueller matrix $\mathbf{M}$ can be normalized so that $\det \mathbf{M_N} = 1$ and $(\mathbf{M_N})_{00} \geq 1$. The set of matrices normalized in this way is isomorphic to the L_+ group, and this one is also homomorphic 2:1 to SL(2C) group, in the way ±A↔L, with $L \in L_+$ [31].

Within the L_+ group can be distinguished two types of transformations [31]: the pure Lorentz ones and the spatial rotations. The former are characterized by symmetric Mueller matrices $\mathbf{K}$ that correspond to partial polarizers, and the last are characterized by orthogonal matrices $\mathbf{R}$ that correspond to retarders.

It is known that a generic element $\mathbf{M}_N$ of the L_+ group can be expressed, in a unique way, as follows [31]

$$\mathbf{M_N} = \mathbf{RK} = \mathbf{K_1 R_1} \qquad \text{(II.107)}$$

Similarly, a generic element A of the SL(2C) group can be expressed, in a unique way, in the form

$$\mathbf{A} = \mathbf{UH} = \mathbf{H_1 U_1} \qquad \text{(II.108)}$$

There are N-type optical systems such as their associated Jones and Mueller matrices, $\mathbf{J}$ and $\mathbf{M}$ respectively, cannot be normalized to have the determinant equal to one because they are singular matrices. Such systems are composed of a set of optical media, in which there is at least a total polarizer.

This statement is based on the fact that, as we will see in the next section, any N-type optical system is equivalent to a certain combination of retarders and polarizers, in such a way that, if their associated matrices $\mathbf{J}$ and $\mathbf{M}$ in the formalism JCF and SMF respectively have zero determinant is because one of the components is a total polarizer.

The optical systems that depolarize, in more or less extent, the light passing through them, have not an associated Jones matrix, and their associated Mueller matrix $\mathbf{M}$ cannot be normalized in such a way that $\mathbf{M}_N$ belong to the L_+ group. Then, we see that only the Mueller matrices associated with non N-type systems and the Mueller matrices associated with systems that contain a total polarizer are out of the L_+ group. Similarly, the Jones matrices corresponding to systems that contain a total polarizer are the only ones that are out of the SL(2C) group.

II.7 Polar decomposition

The expressions (II.107) and (II.108) show a particular case of the polar decomposition theorem for a linear operator [38]. As a consequence of this theorem, any Mueller matrix **M** can be written as follows

$$\mathbf{M} = \mathbf{R}\mathbf{K} = \mathbf{K}_1\mathbf{R}_1 \tag{II.109.a}$$

and any Jones matrix **J** can be written in the form

$$\mathbf{J} = \mathbf{U}\mathbf{H} = \mathbf{H}_1\mathbf{U}_1 \tag{II.109.b}$$

The matrices $\mathbf{K}$, $\mathbf{K}_1$, $\mathbf{H}$, $\mathbf{H}_1$ are always unique, and the matrices $\mathbf{R}$, $\mathbf{R}_1$, $\mathbf{U}$, $\mathbf{U}_1$ are unique except for the case in which **M** and **J** are singular.

II.8. Theorems

The classic works of R.C. Jones [1] states a series of theorems of equivalence, established for N-type media transforming quasi-monochromatic light. In these works, the theorems are proved by means of the matricial calculus introduced by Jones himself. Later, C Whitney [18] generalizes some of these theorems, basing his considerations on the Pauli algebra and the theorem of polar decomposition of a matrix corresponding to a linear operator. Now, we formulate and discuss the most important theorems, including some results that have been established by us [39] as well as other ones that are stated for the first time in this work. The following theorems, except for T11 and T12, are formulated for N-type optical media transforming monochromatic light.

T1.- An optical system that contains a series of any number of retarders (linear, circular or elliptic) is optically equivalent to an elliptic retarder.

T2.- Any elliptic retarder is optically equivalent to a system composed of a sequence of a linear retarder and a rotator.

T3.- Any elliptic retarder is optically equivalent to a serial system composed of two linear retarders (in a non-unique way).

T4.- Any optical system composed of a series of any number of retarders (linear, circular or elliptic) is optically equivalent to a system that contains a sequence of a linear retarder and a rotator [1]

T5.- Any optical system composed of a series of any number of retarders (linear, circular or elliptic) is optically equivalent to a system that contains a series of two linear retarders (in a non-unique way) [18].

The previous theorems can be proved by means of the Rodrigues-Hamilton theorem [18].

T6.- A partial (total) elliptic polarizer is optically equivalent to a system composed of a partial (total) linear polarizer placed between two equal linear retarders whose axes are perpendicular.

T7.- An optical system composed of a series of any number of linear partial polarizers and rotators is optically equivalent to a system composed of a sequence of a linear partial polarizer and a rotator [1].

T8.-The polar decomposition theorem (PDT)

An optical system composed of a series of any number of retarders (linear, circular or elliptic) and partial polarizers (linear, circular or elliptic) is optically equivalent to a serial system composed of an elliptic retarder and an elliptic partial polarizer [18].

In this last theorem we can distinguish between two cases, regarding the nature of the N-type optical system considered. In one case, the system contains a total polarizer, and then the Jones matrix associated to it is singular, and can be written as the product between a Hermitian singular matrix (elliptic total polarizer) and a unitary matrix (retarder), which is non-unique. In the other case, the equivalent system is unique and it is composed of a series of an elliptic partial polarizer and a retarder.

T9.- Equivalence general theorem (EGT)

An optical system composed of a series of any number of retarders (linear, circular or elliptic) and partial polarizers (linear, circular or elliptic) is optically equivalent to a serial system composed of four elements: one partial polarizer between two linear retarders, and a rotator in any of the four possible positions [1].

The two last theorems are formulated for any N-type system. Although the PDT theorem is more synthesized than the EGT, the last one is of great interest because it provides us an equivalent system composed of simple optical media, i.e. circular and linear retarders and linear polarizers.

T10.- Reciprocity theorem of in JCF and CMF formalisms

The Jones matrix associated with an optical system that is passed through by a light beam on a certain direction must be transposed in order to obtain the Jones matrix of the same optical system when it is passed through by a light beam on the opposite direction [1, 3].

T11.- Reciprocity theorem in CVF formalism

The matrix **V** associated, in CVF formalism, with an optical system that is passed through by a light beam on a certain direction, must be transposed in order to obtain the associated matrix of the same optical system, in the same formalism, when it is passed through by a light beam on the opposite direction.

T12.- Theorem of reciprocity in SMF formalism

If an optical system has associated a Mueller matrix **M**, then the Mueller matrix associated with the same optical system when the light beam passes through it in the opposite direction is given by **M'**, according to the expressions (II.76) and (II.77).

In the case of N-type optical media, the proof of the last two theorems is immediate, because, according to the theorem T10, we know that a matrix $\mathbf{J}^T$ corresponds to a Jones matrix **J** if the light beam is passing through in the opposite direction. In section II.6.3 we saw that if the matrices **M** and **V** correspond to a matrix **J**, in SMF and CVF formalisms respectively, the matrices **M'** and $\mathbf{V}^T$ correspond to $\mathbf{J}^T$. In the case of G-type systems, their associated matrices **M** and **V** can be considered as the sum of N-type matrices, in the following form

$$\mathbf{M} = \sum_i \mathbf{M}_i, \qquad \mathbf{V} = \sum_i \mathbf{V}_i \tag{II.110}$$

where $\mathbf{M}_i$ and $\mathbf{V}_i$ are N-type for any i.

If the light passes through on the opposite direction, the corresponding matrices $\mathbf{M}_1$ and $\mathbf{V}_1$ are given by

$$\mathbf{M}_1 = \sum_i \mathbf{M}'_i = \mathbf{M}', \qquad \mathbf{V}_1 = \sum_i \mathbf{V}_i^T = \mathbf{V}^T \tag{II.111}$$

Thus, the theorems T11 and T12 have been demonstrated for the general case of G-type optical media.

T13.- Transcendent rotator theorem (TRT)

An optical system composed of a series of two half wave linear retarders is optically equivalent to a rotator that produces a rotation equal to the double of the angle formed by the fast axes of the half wave plates[39, 17].

T14.- Linear retardation compensator theorem (LRCT)

An optical system composed of a series of three linear retarders, in such a way that the placed at the extremes are equal and whose fast axes are aligned, is optically equivalent to a linear retarder [39].

Chapter III

Properties of the matrices that represent optical media

In this chapter we present the analytic expressions of the elements of a generic N-type Mueller matrix. Each element is expressed as a function of the parameters associated with the equivalent optical systems given by the theorems EGT and PDT. Later we analyze in detail the mathematical restrictions affecting the matrices associated with optical media in the SMF and CVF formalisms. These restrictions are presented as systems of equalities and inequalities; and from these restrictions is finally established a necessary and sufficient condition for an optical medium to be N-type (norm condition). This result is formulated in the SMF, CVF and JCF formalisms. The study of these characteristic properties is useful for the extraction of the physical information contained in the matrices associated with the measured optical media, letting us obtaining theory-experience adjustments as a function of few parameters, and performing the classification of these media in a systematic and simple way.

In the development of this chapter we have preferred to make a rigorous and compact treatment, unifying notations and bringing together results from other authors, which we include with our original contributions in an organized manner. Consequently, we indicate explicitly the contributions from other authors, and the remainder must be understood as original contribution.

III.1. Degree of polarization in the different formalisms

The degree of polarization G of a light beam is defined as the ratio between the intensity of the part of the light that is totally polarized (whatever the state of polarization) and the total intensity.

Any light beam with Stokes vector **S** can be decomposed in the form (II.23), and thus

$$G = \frac{I_P}{I_T} = \frac{I_P}{I_P + I_N} = \frac{\left(s_1^2 + s_2^2 + s_3^2\right)^{1/2}}{s_0} \qquad \text{(III.1)}$$

The expression of G as a function of the elements of the coherency matrix ρ is the following

$$G = \frac{\left(\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 + 4\rho_3\rho_4\right)^{1/2}}{\left(\rho_1 + \rho_2\right)} \qquad \text{(III.2)}$$

or [40]

$$G = \frac{\left[\left(tr\boldsymbol{\rho}\right)^2 - 4\det\boldsymbol{\rho}\right]^{1/2}}{tr\boldsymbol{\rho}} \tag{III.3}$$

Another quantity of interest is the quadratic form *F*, whose relation with *G* has been expressed in (II.27), and whose expression as a function of the coherency matrix $\boldsymbol{\rho}$ is

$$F = 4\det\boldsymbol{\rho} \tag{III.4}$$

A beam of totally polarized light is characterized by the values *G* = 1, *F* = 0, that is

$$s_0^2 = s_1^2 + s_2^2 + s_3^2 \tag{III.5}$$

$$\det\boldsymbol{\rho} = 0 \tag{III.6}$$

Since the JCF only support the representation of totally polarized states, the quantities *G* and *F* are out of the JCF framework.

III.2. Construction of a generic Mueller matrix

III.2.1 Equivalence general theorem

Let us consider a system constituted by a series of N-type optical media arranged successively in the path of the light beam interacting with them. According to the EGT theorem, there is an equivalent system that, in order to be specific, we can suppose in the following order: a rotator R(ω), a linear retarder L(θ_1, δ_1), a partial polarizer P(α, p_1, p_2) and a linear retarder L(θ_2, δ_2).

The Mueller matrix **M** associated with the equivalent system is obtained as the ordered product of the associated matrices as follows

$$\mathbf{M} = \mathbf{M_L}\left(\theta_2,\delta_2\right)\mathbf{M_P}\left(\alpha,p_1,p_2\right)\mathbf{M_L}\left(\theta_1,\delta_1\right)\mathbf{M_R}\left(\omega\right) \tag{III.7}$$

For any Mueller matrix **M**(φ) associated with a generic medium whose polarization axis has an angle φ with the reference X axis, the following properties are fulfilled

$$\mathbf{M}\left(\theta+\varphi\right) = \mathbf{M_R}\left(-\theta\right)\mathbf{M}\left(\varphi\right)\mathbf{M_R}\left(\theta\right) \tag{III.8}$$

$$\mathbf{M}(\alpha+\beta)=\mathbf{M_R}(\alpha)\mathbf{M_R}(\beta) \quad \text{(III.9)}$$

By applying these properties in (III.7), we can write

$$\mathbf{M}=\mathbf{M_R}(\tau_4)\mathbf{M_L}(0,\delta_2)\mathbf{M_R}(\tau_3)\mathbf{M_P}(0,p_1,p_2)\mathbf{M_R}(\tau_2)\mathbf{M_L}(0,\delta_1)\mathbf{M_R}(\tau_1) \quad \text{(III.10.a)}$$

where

$$\tau_1=\theta_1+\omega, \quad \tau_2=\alpha-\theta_1, \quad \tau_3=\theta_2-\alpha, \quad \tau_4=-\theta_2 \quad \text{(III.10.b)}$$

Wherever appropriate, will use the following abbreviated notation in order to simplify mathematical expressions

$$s\equiv\sin\delta, \quad c\equiv\cos\delta, \quad s'\equiv\sin\delta_2, \quad c'\equiv\cos\delta_2 \quad \text{(III.11)}$$

The matrices shown in (III.10.a) have the following generic form [3]

$$\mathbf{M}_\mathrm{R}(\gamma)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos 2\gamma & \sin 2\gamma & 0\\ 0 & -\sin 2\gamma & \cos 2\gamma & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \quad \text{(III.12)}$$

$$\mathbf{M_L}(0,\delta)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos\delta & -\sin\delta\\ 0 & 0 & \sin\delta & \cos\delta\end{pmatrix} \quad \text{(III.13)}$$

$$\mathbf{M_P}(0,p_1,p_2)=\frac{1}{2}\begin{pmatrix}p_1^2+p_2^2 & p_1^2-p_2^2 & 0 & 0\\ p_1^2-p_2^2 & p_1^2+p_2^2 & 0 & 0\\ 0 & 0 & 2p_1p_2 & 0\\ 0 & 0 & 0 & 2p_1p_2\end{pmatrix} \quad \text{(III.14)}$$

Once the product indicated in (III.10.a) has been carried out, we obtain the expressions of the elements m_{ij} of the generic Mueller matrix **M**, as a function of the characteristic parameters of the equivalent system. These expressions are

$$
\begin{aligned}
m_{00} &= q_1 \\
m_{01} &= q_2\left(c_1c_2 - s_1s_2c\right) \\
m_{02} &= q_2\left(s_1c_2 + c_1s_2c\right) \\
m_{03} &= q_2s_2s \\
m_{10} &= q_2\left(c_3c_4 - s_3s_4c'\right) \\
m_{11} &= q_1\left(c_1c_2 - s_1s_2c\right)\left(c_3c_4 - s_3s_4c'\right) - q_3\left(c_1s_2 - s_1s_2c\right)\left(c_4c_3 + c_3s_4c'\right) + q_3s_1s_4ss' \\
m_{12} &= q_1\left(s_1c_2 + c_1s_2c\right)\left(c_3c_4 - s_3s_4c'\right) + q_3\left(-s_1s_2 + c_1c_2c\right)\left(c_4c_3 + c_3s_4c'\right) - q_3c_1s_4ss' \\
m_{13} &= q_1s_2s\left(c_3c_4 - s_3s_4c'\right) + q_3c_2s\left(c_4c_3 + c_3s_4c'\right) + q_3s_4cs' \\
m_{20} &= -q_2\left(s_4c_3 + s_3c_4c'\right) \\
m_{21} &= -q_1\left(c_1c_2 - s_1s_2c\right)\left(c_3s_4 + s_3c_4c'\right) - q_3\left(c_1s_2 + s_1c_2c\right)\left(-s_3s_4 + c_3c_4c'\right) + q_3s_1c_4ss' \\
m_{22} &= -q_1\left(s_1c_2 + c_1s_2c\right)\left(c_3s_4 + s_3c_4c'\right) + q_3\left(-s_1s_2 + c_1c_2c\right)\left(-s_3s_4 + c_3c_4c'\right) - q_3c_1c_4ss' \\
m_{23} &= -q_1s_2s\left(c_3s_4 + s_3c_4c'\right) + q_3c_2s\left(-s_4s_3 + c_3c_4c'\right) + q_3c_4cs' \\
m_{30} &= q_2s_3s' \\
m_{31} &= q_1\left(c_1c_2 - s_1s_2c\right)s_3s' + q_3\left(c_1s_2 + s_1c_2c\right)c_3s' + q_3s_1sc' \\
m_{32} &= q_1\left(s_1c_2 + c_1s_2c\right)s_3s' - q_3\left(-s_1s_2 + c_1c_2c\right)c_3s' - q_3c_1sc' \\
m_{33} &= q_1s_2s_3ss' - q_3c_2c_3ss' + q_3cc'
\end{aligned}
\qquad \text{(III.15.a)}
$$

where

$$
q_1 = \frac{1}{2}\left(p_1^2 + p_2^2\right), \quad q_2 = \frac{1}{2}\left(p_1^2 - p_2^2\right), \quad q_3 = p_1p_2 \qquad \text{(III.15.b)}
$$

III.2.2. Polar decomposition theorem

The PDT theorem implies that the polarization properties of an N-type optical medium are characterized in general by seven independent parameters, four of which correspond to the equivalent partial polarizer and three correspond to the equivalent retarder.

Given an elliptic partial polarizer, its associated Jones matrix $\mathbf{J_{PE}}$ is given by

$$
\mathbf{J_{PE}} = \begin{pmatrix} p_1'\cos^2\nu + p_2'\sin^2\nu & \left(p_1' - p_2'\right)\cos\nu\sin\nu.e^{-i\delta} \\ \left(p_1' - p_2'\right)\cos\nu\sin\nu.e^{i\delta} & p_1'\sin^2\nu + p_2'\cos^2\nu \end{pmatrix} \qquad \text{(III.16.a)}
$$

with

$$c \equiv \cos \nu, \;\; s \equiv \sin \nu \qquad \text{(III.16.b)}$$

where p_1', p_2' are the principal coefficients of the amplitude transmission corresponding to the two invariant orthogonal eigenstates of polarization. These eigenstates are defined by azimuths χ and $\chi + \pi/2$, and ellipticities ω and $-\omega$ respectively, such that

$$\tan 2\chi = \tan 2\nu \cos \delta \qquad \text{(III.17.a)}$$

$$\sin 2\omega = \sin 2\nu \sin \delta \qquad \text{(III.17.b)}$$

The matrix $\mathbf{J_{PE}}$ can be obtained through the theorem T6, which can be applied choosing the orientation of the equivalent linear partial polarizer in such a manner that the axes of the two equivalent retarders are aligned with the axes X and Y of a prefixed Cartesian reference system. Thus

$$\mathbf{J_{PE}} = \mathbf{J_L}\left(0, -\frac{\delta}{2}\right)\mathbf{J_R}(-\nu)\mathbf{J}\left(0, p_1', p_2'\right)\mathbf{J_R}(\nu)\mathbf{J_L}\left(0, \frac{\delta}{2}\right) \qquad \text{(III.18)}$$

Analogously, the Mueller matrix $\mathbf{M_{PE}}$ associated with the same elliptic partial polarizer is obtained as

$$\mathbf{M_{PE}} = \mathbf{M_L}\left(0, -\frac{\delta}{2}\right)\mathbf{M_R}(-\nu)\mathbf{M_P}\left(0, p_1', p_2'\right)\mathbf{M_R}(\nu)\mathbf{M_L}\left(0, \frac{\delta}{2}\right) \qquad \text{(III.19)}$$

On the other hand, according to the theorem T2, the Mueller matrix $\mathbf{M_E}$ associated with an elliptic retarder can be written in the form

$$\mathbf{M_E} = \mathbf{M_L}\left(\alpha, \delta'\right)\mathbf{M_R}(\beta) \qquad \text{(III.20)}$$

According to the PDT theorem, every N-type Mueller matrix $\mathbf{M}$ can be written as follows

$$\mathbf{M} = \mathbf{M_{PE}}\mathbf{M_E} \qquad \text{(III.21)}$$

According to the theorem T4, there are two matrices $\mathbf{M_L}$ (ζ, Δ_1) and $\mathbf{M_R}(\gamma)$ such as

$$\mathbf{M_R}(\nu)\mathbf{M_L}\left(0, \frac{\delta}{2}\right)\mathbf{M_L}\left(\alpha, \delta'\right)\mathbf{M_R}(\beta) = \mathbf{M_L}\left(\xi, \Delta_1\right)\mathbf{M_R}(\gamma) \qquad \text{(III.22)}$$

The last expressions let us write

$$\mathbf{M} = \mathbf{M_L}\left(0, -\Delta_2\right)\mathbf{M_R}(-\nu)\mathbf{M_P}\left(0, p_1, p_2\right)\mathbf{M_L}\left(\xi, \Delta_1\right)\mathbf{M_R}(\nu) \qquad \text{(III.23.a)}$$

where

$$\Delta_2 \equiv \delta/2 \qquad \text{(III.23.b)}$$

or

$$\mathbf{M} = \mathbf{M_L}\left(0, -\Delta_2\right)\mathbf{M_R}(-\nu)\mathbf{M_P}\left(0, p_1, p_2\right)\mathbf{M_R}(-\xi)\mathbf{M_L}\left(0, \Delta_1\right)\mathbf{M_R}(\xi + \gamma) \qquad \text{(III.24)}$$

By comparing the expressions (III.10) and (III.24) we see that (III.24) is a particular case of (III.10), with the following correspondence between the parameters

$$\begin{array}{l} \tau_1 = \xi + \gamma, \quad \tau_2 = -\xi, \quad \tau_3 = -\nu, \quad \tau_4 = 0, \\ \Delta_1 = \delta_1, \quad \Delta_2 = \delta_2, \quad p_1' = p_1, \quad p_2' = p_2 \end{array} \qquad \text{(III.25)}$$

Since the equivalent system given by the expression (III.10) depends on 8 parameters (τ_1, τ_2, τ_3, τ_4, δ_1, δ_1, p_1, p_2), seven of which are independent, we have the freedom of choosing an arbitrary value for τ_4. A convenient choice, in order to simplify subsequent calculations is $\tau_4 = 0$.

By writing explicitly the expression (III.24) for the elements m_{ij} of the generic matrix **M** we obtain

$$\begin{aligned}
m_{00} &= q_1' \\
m_{01} &= q_2'\left(c_1'c_2' - s_1's_2'c''\right) \\
m_{02} &= q_2'\left(s_1'c_2' + c_1's_2'c''\right) \\
m_{03} &= q_2's_2's''
\end{aligned}$$

$$\begin{aligned}
m_{10} &= q_2'c_3' \\
m_{11} &= q_1'\left(c_1'c_2' - s_1's_2'c''\right) - q_3'\left(c_1's_2' + s_1'c_2'c''\right)s_3' \\
m_{12} &= q_1'\left(s_1'c_2' + c_1's_2'c''\right) - q_3'\left(s_1's_2' - c_1'c_2'c''\right)s_3' \\
m_{13} &= q_1's_2'c_3's'' + q_3'c_2's_3's'' \\
m_{20} &= -q_2's_3'c''' \\
m_{21} &= -q_1'\left(c_1'c_2' - s_1's_2'c''\right)s_3'c''' - q_3'\left(c_1's_2' + s_1'c_2'c''\right)c_3'c''' + q_3's_1's''s''' \\
m_{22} &= -q_1'\left(s_1'c_2' + c_1's_2'c''\right)s_3'c''' - q_3'\left(s_1's_2' - c_1'c_2'c''\right)c_3'c''' - q_3'c_1's''s''' \\
m_{23} &= -q_1's_2's_3's''c''' + q_3'c_2c_3s''c''' + q_3'c''s'''
\end{aligned} \qquad \text{(III.26.a)}$$

$$m_{30} = q_2' s_3' c'''$$
$$m_{31} = q_1'\left(c_1'c_2' - s_1's_2'c''\right)s_3's''' + q_3'\left(c_1's_2' + s_1'c_2'c''\right)c_3's''' + q_3's_1's'c'''$$
$$m_{32} = -q_1'\left(s_1'c_2' + c_1's_2'c''\right)s_3's''' + q_3'\left(s_1's_2' - c_1'c_2'c''\right)c_3's''' - q_3'c_1's''c'''$$
$$m_{33} = q_1's_2's_3's''s''' - q_3'c_2'c_3's''s''' + q_3'c''c'''$$

where

$$\begin{aligned} &c_1' = \cos 2(\xi+\gamma), \quad c_2' = \cos(-2\xi), \quad c_2' = \cos(-2\nu), \\ &s_1' = \sin 2(\xi+\gamma), \quad s_2' = \sin(-2\xi), \quad s_2' = \mathrm{c}\sin(-2\nu), \\ &c'' = \cos\Delta_1, \quad c''' = \cos\Delta_2 = \cos\left(-\delta/2\right), \\ &s'' = \sin\Delta_1, \quad s''' = \sin\Delta_2 = \sin\left(-\delta/2\right) \end{aligned} \tag{III.26.b}$$

The advantage of applying the PDT theorem, instead of the EGT theorem, is on the one hand the obtainment of all the elements of a generic Mueller matrix as functions of a minimum set of independent parameters (seven) and, on the other hand, the PDT theorem let us synthesize the equivalent system with only two optical media (a polarizer and a retarder) instead of four (as in the EGT theorem). However, when we write the generic matrix obtained by means of the PDT theorem as a function of simpler matrices, associated with linear retarders, rotators and linear polarizers, the equivalent system is composed of five simple elements (two linear retarders, two rotators and a linear partial polarizer).

From the expressions (III.15) and (III.26), it is easy to obtain the following relations

$$q_1' = q_1 = m_{00} \tag{III.27.a}$$

$$q_2'^2 = q_2^2 = \left(m_{01}^2 + m_{02}^2 + m_{03}^2\right) = \left(m_{10}^2 + m_{20}^2 + m_{30}^2\right) \tag{III.27.b}$$

and, thus

$$p_1' = p_1, \quad p_2' = p_2 \tag{III.28}$$

III.3. Classification of the N-type Mueller matrices

According to (III.26), the seven parameters that characterize the equivalent system are Δ_1, Δ_2, p_1, p_2, ν, ξ, γ. In order to obtain these parameters as functions of the elements m_{ij}, we can distinguish among three cases

CASE 1: $\left(m_{01}^2+m_{02}^2+m_{03}^2\right)^{1/2}=0$

In this case $q_2 = 0$, and according to (III.27) the elements of the first row and first column, except m_{00}, are zero. The matrix corresponds to an elliptic retarder, whose generic form is [30]

$$\mathbf{M}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A_1^2-A_2^2-A_3^2+A_4^2 & 2\left(A_1A_2+A_3A_4\right) & -2\left(A_1A_3+A_2A_4\right) \\ 0 & 2\left(A_1A_2-A_3A_4\right) & -A_1^2+A_2^2-A_3^2+A_4^2 & 2\left(A_1A_4-A_2A_3\right) \\ 0 & -2\left(A_1A_3-A_2A_4\right) & -2\left(A_1A_4+A_2A_3\right) & -A_1^2-A_2^2+A_3^2+A_4^2 \end{pmatrix} \tag{III.29.a}$$

where

$$\begin{aligned} A_1 &= \cos 2\omega \cos 2\psi \sin\left(\Delta/2\right) \\ A_2 &= \cos 2\omega \sin 2\psi \sin\left(\Delta/2\right) \\ A_3 &= \sin 2\omega \sin\left(\Delta/2\right) \\ A_4 &= \cos\left(\Delta/2\right) \end{aligned} \tag{III.29.b}$$

being ψ the azimuth, ω the ellipticity of its two orthogonal elliptic eigenstates, and Δ the retardation introduced between them.

The parameters Δ, ω, ψ are obtained as

$$\cos^2\left(\Delta/2\right)=\frac{1}{4}\left(\mathrm{tr}\mathbf{M}\right)=\frac{1}{4}\left(m_{00}+m_{11}+m_{22}+m_{33}\right) \tag{III.30.a}$$

$$\sin 2\omega=\frac{\left(m_{12}-m_{21}\right)}{2\sin\Delta} \tag{III.30.b}$$

$$\sin 2\psi=\frac{\left(m_{21}-m_{12}\right)}{2\cos 2\omega \sin\Delta} \tag{III.30.c}$$

It is worth mentioning that when $\omega=\pm\pi/4$, the matrix **M** corresponds to a rotator; and if $\omega=0$, then it corresponds to a linear retarder.

CASE 2: $0<\left(m_{01}^2+m_{02}^2+m_{03}^2\right)^{1/2}\le m_{00}$ and $\mathbf{M}^T=\mathbf{M}$

The matrix **M** corresponds to an elliptic partial polarizer with the generic form

$$\mathbf{M} = \begin{pmatrix} q_1 & q_2 c_\nu & q_2 s_\nu c_\delta & -q_2 s_\nu s_\delta \\ q_2 c_\nu & q_1 c_\nu^2 + q_3 s_\nu^2 & c_\nu s_\nu c_\delta (q_1 - q_3) & -c_\nu s_\nu s_\delta (q_1 - q_3) \\ q_2 s_\nu c_\delta & c_\nu s_\nu c_\delta (q_1 - q_3) & c_\delta^2 (q_1 s_\nu^2 + q_3 c_\nu^2) + q_3 s_\delta^2 & -c_\delta s_\delta (q_1 s_\nu^2 + q_3 c_\nu^2 - q_3) \\ -q_2 s_\nu s_\delta & -c_\nu s_\nu s_\delta (q_1 - q_3) & -c_\delta s_\delta (q_1 s_\nu^2 + q_3 c_\nu^2 - q_3) & (q_1 s_\nu^2 + q_3 c_\nu^2) s_\delta^2 + q_3 c_\delta^2 \end{pmatrix} \tag{III.31.a}$$

where

$$\begin{gathered} q_1 = \frac{1}{2}\left(p_1^2 + p_2^2\right), \quad q_2 = \frac{1}{2}\left(p_1^2 - p_2^2\right), \quad q_3 = p_1 p_2, \\ c_\nu = \cos 2\nu, \quad s_\nu = \sin 2\nu, \quad c_\delta = \cos\left(-\delta/2\right), \quad s_\delta = \sin\left(-\delta/2\right) \end{gathered} \tag{III.31.b}$$

The meaning of p_1, p_2, ν, δ is the same as of p_1', p_2', ν, δ in the expression (III.16). These parameters are given by

$$\tan\left(\delta/2\right) = \frac{m_{30}}{m_{20}} = \frac{m_{03}}{m_{02}} = \frac{m_{13}}{m_{12}} = \frac{m_{31}}{m_{21}} \tag{III.32.a}$$

$$\tan\left(2\nu\right) = \frac{m_{20}}{m_{10}} \cos\left(\delta/2\right) = -\frac{m_{30}}{m_{10}} \sin\left(\delta/2\right) \tag{III.32.b}$$

$$p_1^2 = \left(m_{00} + \frac{m_{10}}{\cos 2\nu} \right) \tag{III.32.c}$$

$$p_2^2 = \left(m_{00} - \frac{m_{10}}{\cos 2\nu} \right) \tag{III.32.d}$$

CASE 3: $0 < \left(m_{01}^2 + m_{02}^2 + m_{03}^2\right)^{1/2} \leq m_{00}$ and $\mathbf{M}^T \neq \mathbf{M}$

From (III.26) we see that

$$q_1 \neq 0, \quad q_2 \neq 0 \tag{III.33}$$

and, moreover

$$q_2 < q_1 \tag{III.34}$$

so that

$$p_1 \neq p_2, \quad p_1 \neq 0, \quad p_2 \neq 0 \tag{III.35}$$

The matrix corresponds to a system with simultaneous properties of retardation and partial polarization. The equivalent parameters are

$$\tan\left(\delta/2\right) = m_{30}/m_{20} \tag{III.36.a}$$

$$\tan\left(2\nu\right) = \frac{\left(m_{30}^2 + m_{20}^2\right)^{1/2}}{m_{10}} \tag{III.36.b}$$

$$p_1^2 = m_{00} + \left(m_{01}^2 + m_{02}^2 + m_{03}^2\right)^{1/2} \tag{III.36.c}$$

$$p_2^2 = m_{00} - \left(m_{01}^2 + m_{02}^2 + m_{03}^2\right)^{1/2} \tag{III.36.d}$$

$$\cot\left(2\xi\right) = \frac{1}{\sin 2\nu}\left(\frac{q_2 m_{13}}{q_3 m_{03}} - \frac{q_1}{q_3}\cos 2\nu\right) \tag{III.36.e}$$

$$\sin\Delta_1 = -\frac{m_{03}}{q_2 \sin 2\xi} \tag{III.36.f}$$

$$\sin 2(\xi+\gamma) = \frac{m_{20}}{m_{30} q_3 \sin\Delta_1}\left[\frac{q_1 m_{01}}{q_2 \sin 2\nu} + m_{11}\cot(2\nu) - \frac{m_{21}}{\cos\left(\delta/2\right)}\right] \tag{III.36.g}$$

The expressions (III.36) give us the parameters that correspond to an equivalent system T, whose elements are

$$\mathbf{T} \equiv \mathbf{L}\left(0,\delta/2\right)\mathbf{R}\left(-\nu\right)\mathbf{P}\left(0,p_1,p_2\right)\mathbf{L}\left(\xi,\Delta_1\right)\mathbf{R}\left(\gamma\right) \tag{III.37}$$

The parameters p_1, p_2, ν, δ characterize the equivalent elliptic polarizer mentioned in the PDT theorem.

Given a Mueller matrix, it is interesting now to obtain the parameters that characterize the equivalent elliptic retarder, in order to determine the equivalent polarizer and the equivalent retarder.

The expression (III.22) can be written in the form

$$\begin{aligned} &\mathbf{M_R}\left(\nu\right)\mathbf{M_L}\left(0,\delta/2\right)\mathbf{M_L}\left(\alpha,\delta'\right)\mathbf{M_R}\left(\beta\right) = \mathbf{M_L}\left(\xi,\Delta_1\right)\mathbf{M_R}\left(\gamma\right) = \\ &= \mathbf{M_R}\left(\nu\right)\mathbf{M_R}\left(-\nu\right)\mathbf{M_L}\left(\xi,\Delta_1\right)\mathbf{M_R}\left(\nu\right)\mathbf{M_R}\left(-\nu\right)\mathbf{M_R}\left(-\beta\right)\mathbf{M_R}\left(\beta\right) = \\ &= \mathbf{M_R}\left(\nu\right)\mathbf{M_L}\left(\eta,\Delta_1\right)\mathbf{M_R}\left(\mu\right)\mathbf{M_R}\left(\beta\right) \end{aligned} \tag{III.38.a}$$

with

$$\eta = \nu + \xi, \quad \mu = \gamma - \nu - \beta \tag{III.38.b}$$

from which we obtain the equality

$$\mathbf{M_L}\left(0, \delta/2\right)\mathbf{M_L}\left(\alpha, \delta'\right) = \mathbf{M_L}\left(\eta, \Delta_1\right)\mathbf{M_R}\left(\mu\right) \tag{III.39}$$

or, in JCF formalism

$$\mathbf{J_L}\left(0, \delta/2\right)\mathbf{J_L}\left(\alpha, \delta'\right) = \mathbf{J_L}\left(\eta, \Delta_1\right)\mathbf{J_L}\left(\mu\right) \tag{III.40}$$

First of all we will try to obtain the unknown parameters α, δ', β that characterize the linear retarder L(α, δ') and the rotator R(β) as functions of the known parameters. Later, we will write the Mueller matrix **M** as a product of the matrices associated with an elliptic partial polarizer, a linear retarder and a rotator with known characteristics. Finally, we will obtain the parameters that characterize the elliptic retarder equivalent to the system composed of the linear retarder and the rotator.

By performing the matricial product indicated in each member of (III.40), we obtain

$$\mathbf{J} = \begin{pmatrix} c^2 e^{i(q+t)} + s^2 e^{i(t-q)} & sc\left[e^{i(q+t)} - e^{i(t-q)}\right] \\ sc\left[e^{-i(t-q)} - e^{-i(t+q)}\right] & s^2 e^{i(q-t)} + c^2 e^{-i(q+t)} \end{pmatrix} \tag{III.41.a}$$

with

$$c = \cos\alpha, \quad s = \sin\alpha, \quad t = \delta/4, \quad q = \delta'/4 \tag{III.41.b}$$

and, on the other hand,

$$\mathbf{J}' = \mathbf{J_L}\left(\eta, \Delta\right)\mathbf{J_R}\left(\mu\right)\begin{pmatrix} c_2\left(c_1^2 e^{i\tau} + s_1^2 e^{-i\tau}\right) - 2is_1 s_2 c_1 \sin\tau & s_2\left(c_1^2 e^{i\tau} + s_1^2 e^{-i\tau}\right) + 2is_1 c_2 c_1 \sin\tau \\ -s_2\left(s_1^2 e^{i\tau} + c_1^2 e^{-i\tau}\right) + 2is_1 c_2 c_1 \sin\tau & c_2\left(s_1^2 e^{i\tau} + c_1^2 e^{-i\tau}\right) - 2is_1 s_2 c_1 \sin\tau \end{pmatrix} \tag{III.42.a}$$

with

$$\begin{aligned} c_1 &= \cos\eta, \quad s_1 = \sin\eta, \\ c_2 &= \cos\mu, \quad s_2 = \sin\mu, \\ \nu &= \Delta_1/2. \end{aligned} \tag{III.42.b}$$

The equality **J** = **J'** is equivalent to

$$J_1 + J_2 = J_1' + J_2' \tag{III.43.a}$$

$$J_1 - J_2 = J_1' - J_2' \tag{III.43.b}$$

$$J_3 + J_4 = J_3' + J' \tag{III.43.c}$$

$$J_3 - J_4 = J_3' - J_4' \tag{III.43.d}$$

from which we obtain, after some simple operations

$$\cos\mu\cos\tau = \cos q\cos t - \cos 2\alpha\sin q\sin t \tag{III.44.a}$$

$$\sin(\mu - 2\nu)\sin\tau = \sin 2\alpha\sin q\cos t \tag{III.44.b}$$

$$\cos(\mu - 2\nu)\sin\tau = \cos q\sin t + \cos 2\alpha\sin q\cos t \tag{III.44.c}$$

$$\sin\mu\cos\tau = -\sin 2\alpha\sin q\sin t \tag{III.44.d}$$

Working out the unknown parameters in (III.44)

$$\cot\mu = \frac{\cos 2\eta - \cot t\cot\tau}{\sin 2\eta} \tag{III.45.a}$$

$$\beta = \gamma - \nu - \mu \tag{III.45.b}$$

$$\cos(\delta'/2) = \cos(\delta/4)\cos(\Delta_1/4)\cos\mu + \sin(\delta/4)\sin(\Delta_1/4)\cos(\mu - 2\eta) \tag{III.45.c}$$

$$\sin 2\alpha = \frac{\sin\mu\cos(\Delta_1/2)}{\sin(\delta'/2)\sin(\delta/4)} \tag{III.45.d}$$

$$\cos 2\alpha = \frac{\cos(\delta'/2)\cos(\delta/4) - \cos(\Delta_1/2)\cos\mu}{\sin(\delta'/2)\sin(\delta/4)} \tag{III.45.e}$$

The system composed of the equivalent linear retarder L(α, δ') and the rotator R(β), is equivalent to a certain elliptic retarder with orthogonal eigenstates of polarization with azimuth χ_1, ellipticity ψ_1 and a retardation Δ between them.

As seen in (II.7), there are two parameters σ, τ such as

$$\tan 2\chi_1 = \tan 2\sigma\cos\tau \tag{III.46.a}$$

$$\sin 2\psi_1 = \sin 2\sigma\sin\tau \tag{III.46.b}$$

Taking into account the equality between the Jones matrix associated with this elliptic retarder and the matrix **J''** ≡ $\mathbf{J_L}(\alpha, \delta')\,\mathbf{J_R}(\beta)$, and operating in a similar way than before, we finally obtain

$$\cos(\Delta/2) = \cos(\delta'/2)\cos\beta \qquad \text{(III.47.a)}$$

$$\cos 2\sigma = \frac{\sin(\delta'/2)\cos(\beta - 2\alpha)}{\sin(\Delta/2)} \qquad \text{(III.47.b)}$$

$$\sin\tau = \frac{\cos(\delta'/2)\sin\beta}{\sin 2\sigma \sin(\Delta/2)} \qquad \text{(III.47.c)}$$

$$\cos\tau = \frac{\sin(\delta'/2)\sin(\beta + 2\alpha)}{\sin 2\sigma \sin(\Delta/2)} \qquad \text{(III.47.d)}$$

III.4. Restrictive relations in a Mueller matrix

As we have seen, the characteristics of an N-type optical system are given, in general, by seven independent parameters. This implies that there must be a set of nine restrictions among the elements of any N-type Mueller matrix. An N-type optical medium is characterized by the fact that if a beam of totally polarized light passes through it, the emerging beam must also be totally polarized. We impose now this condition in a matricial way in the SMF formalism. Let **M** be the Mueller matrix associated with the system, and **S**, **S'**, the Stokes vectors corresponding to the incident and emerging light beams respectively. These vectors are related as follows

$$s'_i = \sum_{j=0}^{3} m_{ij} s_j \qquad i = 0,1,2,3 \qquad \text{(III.48)}$$

By squaring this expression we obtain

$$s_i'^2 = \sum_{j=0}^{3} m_{ij}^2 s_j^2 + \sum_{\substack{l,k=0 \\ l\neq k}}^{3} m_{il} m_{ik} s_l s_k \qquad \text{(III.49)}$$

The following condition for the vector **S'** to correspond to a totally polarized light beam

$$s_0'^2 = s_1'^2 + s_2'^2 + s_3'^2 \qquad \text{(III.50)}$$

let us write

$$\sum_{j=0}^{3} m_{0j}^2 s_j^2 + \sum_{\substack{l,k=0 \\ l\neq k}}^{3} m_{0l} m_{0k} s_l s_k = \sum_{i=1}^{3}\left(\sum_{j=0}^{3} m_{ij}^2 s_j^2 + \sum_{\substack{l,k=0 \\ l\neq k}}^{3} m_{il} m_{ik} s_l s_k \right) \qquad \text{(III.51)}$$

The relation (III.51) must be fulfilled for any Stokes **S** vector corresponding to a beam of totally polarized light

$$s_0^2 = s_1^2 + s_2^2 + s_3^2 \qquad \text{(III.52)}$$

In particular, the relation (III.51) is fulfilled for the following Stokes vectors

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \qquad \text{(III.53)}$$

which lead to the relations

$$\left(m_{01}+m_{00}\right)^2 = \left(m_{11}+m_{10}\right)^2 + \left(m_{21}+m_{20}\right)^2 + \left(m_{31}+m_{30}\right)^2 \qquad \text{(III.54.a)}$$

$$\left(m_{01}-m_{00}\right)^2 = \left(m_{11}-m_{10}\right)^2 + \left(m_{21}-m_{20}\right)^2 + \left(m_{31}-m_{30}\right)^2 \qquad \text{(III.54.b)}$$

$$\left(m_{02}+m_{00}\right)^2 = \left(m_{12}+m_{10}\right)^2 + \left(m_{20}+m_{22}\right)^2 + \left(m_{32}+m_{30}\right)^2 \qquad \text{(III.55.a)}$$

$$\left(m_{02}-m_{00}\right)^2 = \left(m_{12}-m_{10}\right)^2 + \left(m_{22}-m_{20}\right)^2 + \left(m_{32}-m_{30}\right)^2 \qquad \text{(III.55.b)}$$

$$\left(m_{03}+m_{00}\right)^2 = \left(m_{13}+m_{10}\right)^2 + \left(m_{23}+m_{20}\right)^2 + \left(m_{33}+m_{30}\right)^2 \qquad \text{(III.56.a)}$$

$$\left(m_{03}-m_{00}\right)^2 = \left(m_{13}-m_{10}\right)^2 + \left(m_{23}-m_{20}\right)^2 + \left(m_{33}-m_{30}\right)^2 \qquad \text{(III.56.b)}$$

By adding respectively the pairs (III.54), (III.55) and (III.56) we obtain

$$m_{01}^2 + m_{00}^2 = m_{11}^2 + m_{21}^2 + m_{31}^2 + m_{10}^2 + m_{20}^2 + m_{30}^2 \qquad \text{(III.57.a)}$$

$$m_{02}^2 + m_{00}^2 = m_{12}^2 + m_{22}^2 + m_{32}^2 + m_{10}^2 + m_{20}^2 + m_{30}^2 \qquad \text{(III.57.b)}$$

$$m_{03}^2 + m_{00}^2 = m_{13}^2 + m_{23}^2 + m_{33}^2 + m_{10}^2 + m_{20}^2 + m_{30}^2 \qquad \text{(III.57.c)}$$

and thus

$$m_{01}m_{00} = m_{11}m_{10} + m_{21}m_{20} + m_{31}m_{30} \quad \text{(III.58.a)}$$

$$m_{02}m_{00} = m_{12}m_{10} + m_{22}m_{20} + m_{32}m_{30} \quad \text{(III.58.b)}$$

$$m_{03}m_{00} = m_{13}m_{10} + m_{23}m_{20} + m_{33}m_{30} \quad \text{(III.58.c)}$$

From (III.57), (III.58) and (III.51) we deduce

$$m_{01}m_{02} = m_{11}m_{12} + m_{21}m_{22} + m_{31}m_{32} \quad \text{(III.59.a)}$$

$$m_{01}m_{03} = m_{11}m_{13} + m_{21}m_{23} + m_{31}m_{33} \quad \text{(III.59.b)}$$

$$m_{02}m_{02} = m_{12}m_{13} + m_{22}m_{23} + m_{32}m_{33} \quad \text{(III.59.c)}$$

The set of relations constituted by (III.57) together with (III.58) is equivalent to (III.54). By adding the relations (III.59) to these sets, we obtain two systems of restrictive relations among the elements m_{ij}. We call R_1 to the system of equalities composed of (III.54), (III.55), (III.56) and (III.59); and R_2 to the composed of (III.57), (III.58) and (III.59).

The systems R_1 and R_2 are equivalent, and they express the restrictions in the **M** matrix in two different ways. Afterwards we will see other systems of restrictions that are equivalent to R_1 and R_2. The usefulness of the study of different kinds of presentations for the restrictions consists in the fact that, as we will see, this let us easily obtain interesting results that otherwise would be masked by the mathematical complexity of the expressions.

Now, let us consider a Mueller matrix associated with a G-type optical medium, which can even produce depolarization. Then, the unique condition that must be fulfilled is

$$s_0'^2 \geq s_1'^2 + s_2'^2 + s_3'^2 \quad \text{(III.60)}$$

and thus, taking into account (III.59), we obtain

$$\sum_{j=0}^{3} m_{0j}^2 s_j^2 + \sum_{\substack{l,k=0 \\ l\neq k}}^{3} m_{0l}m_{0k}s_l s_k \geq \sum_{i=1}^{3}\left(\sum_{j=0}^{3} m_{ij}^2 s_j^2 + \sum_{\substack{l,k=0 \\ l\neq k}}^{3} m_{il}m_{ik}s_l s_k \right) \quad \text{(III.61)}$$

The inequality (III.61) is fulfilled for any Stokes vector **S**, and in particular, for the vectors in (III.53), which can be taken to (III.61) in order to give the following inequalities

$$\left(m_{01} + m_{00}\right)^2 \geq \left(m_{11} + m_{10}\right)^2 + \left(m_{21} + m_{20}\right)^2 + \left(m_{31} + m_{30}\right)^2 \quad \text{(III.62.a)}$$

$$\left(m_{01}-m_{00}\right)^2 \geq \left(m_{11}-m_{10}\right)^2+\left(m_{21}-m_{20}\right)^2+\left(m_{31}-m_{30}\right)^2 \quad \text{(III.62.b)}$$

$$\left(m_{02}+m_{00}\right)^2 \geq \left(m_{12}+m_{10}\right)^2+\left(m_{20}+m_{22}\right)^2+\left(m_{32}+m_{30}\right)^2 \quad \text{(III.62.c)}$$

$$\left(m_{02}-m_{00}\right)^2 \geq \left(m_{12}-m_{10}\right)^2+\left(m_{22}-m_{20}\right)^2+\left(m_{32}-m_{30}\right)^2 \quad \text{(III.62.d)}$$

$$\left(m_{03}-m_{00}\right)^2 \geq \left(m_{13}-m_{10}\right)^2+\left(m_{23}-m_{20}\right)^2+\left(m_{33}-m_{30}\right)^2 \quad \text{(III.62.f)}$$

The inequalities that correspond to the equalities (III.57) are

$$m_{01}^2+m_{00}^2 \geq m_{11}^2+m_{21}^2+m_{31}^2+m_{10}^2+m_{20}^2+m_{30}^2 \quad \text{(III.63.a)}$$

$$m_{02}^2+m_{00}^2 \geq m_{12}^2+m_{22}^2+m_{32}^2+m_{10}^2+m_{20}^2+m_{30}^2 \quad \text{(III.63.b)}$$

$$m_{03}^2+m_{00}^2 \geq m_{13}^2+m_{23}^2+m_{33}^2+m_{10}^2+m_{20}^2+m_{30}^2 \quad \text{(III.63.c)}$$

The system R_2 can be obtained by other way. Let us consider a light beam with Stokes vector **S** and coherency matrix ρ, which passes through an N-type optical system whose associated matrices in the formalisms SMF and JCF are **M** and **J** respectively. The emerging light beam is characterized by a Stokes vector **S'** = **MS**, and a coherency matrix ρ' = **J**ρ**J**$^{+}$, from which we deduce

$$\det \boldsymbol{\rho}' = \left|\det \mathbf{J}\right|^2 \det \boldsymbol{\rho} \quad \text{(III.64)}$$

or, taking into account (III.4)

$$F' = \left|\det \mathbf{J}\right|^2 F \quad \text{(III.65)}$$

being

$$F = s_0^2+s_1^2+s_2^2+s_3^2, \quad F' = s_0'^2+s_1'^2+s_2'^2+s_3'^2 \quad \text{(III.66)}$$

The quadratic form *F*, associated with the Stokes vector **S**, can be written as

$$F = \mathbf{s}^T \mathbf{g} \mathbf{s} \quad \text{(III.67)}$$

where **g** is the matrix

$$\mathbf{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{III.68}$$

and $\mathbf{S}^T$ is the transposed row vector of the column vector **S**. Taking into account that

$$\mathbf{s}'^T = \mathbf{s}^T \mathbf{M}^T \tag{III.69}$$

we can write

$$F' = \mathbf{s}'^T \mathbf{g} \mathbf{s}' = \mathbf{s}^T \mathbf{M}^T \mathbf{g} \mathbf{M} \mathbf{s} \tag{III.70}$$

From (III.71) and (III.65) we obtain the relation

$$\mathbf{s}^T \mathbf{M}^T \mathbf{g} \mathbf{M} \mathbf{s} = \left|\det \mathbf{J}\right|^2 \mathbf{s}^T \mathbf{g} \mathbf{s} \tag{III.71}$$

that must be fulfilled for any Stokes vector **S** and, thus [41]

$$\mathbf{M}^T \mathbf{g} \mathbf{M} = \left|\det \mathbf{J}\right|^2 \mathbf{g} \tag{III.72}$$

By writing (III.72) as a function of the elements m_{ij}, and eliminating $|\det\mathbf{J}|^2$, we obtain the new system R_2.

This last development is similar to the presented by R. Barakat [14] in a recent article, which by contrast is carried out on the basis of considerations about the Lorentz ortochronous L_+ group. This obliges us to avoid singular matrices, which correspond to systems including total polarizers, because they cannot be normalized in order to belong to L_+ group.

Because of the theorem T12, we know that given a Mueller matrix **M**, the matrix **M'** given by (II.76) and (II.77) is also a Mueller matrix. Any expression that has been established for **M** is also valid for **M'**, and new relations result among the elements m_{ij}, which can be obtained from the relations seen before by transposing the subscripts of all elements. So, in an N-type matrix **M** case, **M'** is also N-type and the new relations are

$$\left(m_{i0} + m_{00}\right)^2 = \left(m_{i1} + m_{01}\right)^2 + \left(m_{i2} + m_{02}\right)^2 + \left(m_{i3} + m_{03}\right)^2 \tag{III.73.a}$$

$$\left(m_{i0} - m_{00}\right)^2 = \left(m_{i1} - m_{01}\right)^2 + \left(m_{i2} - m_{02}\right)^2 + \left(m_{i3} - m_{03}\right)^2 \tag{III.73.b}$$

$$m_{i0}^2 + m_{00}^2 = m_{i1}^2 + m_{i2}^2 + m_{i3}^2 + m_{01}^2 + m_{02}^2 + m_{03}^2 \tag{III.74}$$

$$m_{i0}m_{00} = m_{i1}m_{01} + m_{i2}m_{02} + m_{i3}m_{03}, \quad i = 1,2,3 \tag{III.75}$$

with i = 1, 2, 3;

$$m_{i0}m_{j0} = m_{i1}m_{j1} + m_{i2}m_{j2} + m_{i3}m_{j3}, \quad i, j = 1,2,3; i \neq j \tag{III.76}$$

with i, j = 1, 2, 3; $i \neq j$.

In a G-type **M** matrix case, **M'** is G-type and we have the inequalities

$$\left(m_{i0} + m_{00}\right)^2 \geq \left(m_{i1} + m_{01}\right)^2 + \left(m_{i2} + m_{02}\right)^2 + \left(m_{i3} + m_{03}\right)^2 \tag{III.77.a}$$

$$\left(m_{i0} - m_{00}\right)^2 \geq \left(m_{i1} - m_{01}\right)^2 + \left(m_{i2} - m_{02}\right)^2 + \left(m_{i3} - m_{03}\right)^2 \tag{III.77.b}$$

$$m_{i0}^2 + m_{00}^2 \geq m_{i1}^2 + m_{i2}^2 + m_{i3}^2 + m_{01}^2 + m_{02}^2 + m_{03}^2, \quad i = 1,2,3 \tag{III.78}$$

with i = 1, 2, 3.

It is worth mentioning the fact that the system of inequalities formed by (III.63) together with (III.78), and the systems (III.62) together with (III.77) are totally equivalent. Thus, a set of six inequalities among the elements of a G-type Mueller matrix corresponds to the set of nine inequalities among the elements of an N-type Mueller matrix.

The new relations obtained from **M'** also correspond to the matrix $\mathbf{M}^T$, what indicates to us that if **M** is a Mueller matrix, $\mathbf{M}^T$ is a Mueller matrix of the same type.

The equality (III.72) can be written by replacing **M** with **M'**, or $\mathbf{M}^T$, obtaining respectively

$$\mathbf{M'}^T \mathbf{g} \mathbf{M'} = \left|\det \mathbf{J}^T\right|^2 \mathbf{g} \tag{III.79}$$

$$\mathbf{M} \mathbf{g} \mathbf{M}^T = \left|\det \mathbf{J}^T\right|^2 \mathbf{g} \tag{III.80}$$

and taking into account that

$$\left|\det \mathbf{J}\right| = \left|\det \mathbf{J}^T\right| = \left|\det \mathbf{J}^+\right| \tag{III.81}$$

we obtain the condition

$$\mathbf{M}^T\mathbf{g}\mathbf{M} = \mathbf{M}\mathbf{g}\mathbf{M}^T = \mathbf{M}'^T\mathbf{g}\mathbf{M}' = \left|\det \mathbf{J}^T\right|^2 \mathbf{g} \tag{III.82}$$

from which all restrictive inequalities found so far can be deduced.

Next we will obtain another set of nine inequalities among the elements of an N-type Mueller matrix, which are different, although equivalent, from the ones seen before.

The elements of a Jones matrix **J** can be written by means of the notation given in (II.10) and (II.11) as

$$\mathbf{J} = \begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \end{pmatrix} \tag{III.83.a}$$

We can write it in modulus-argument form as follows

$$A_k \equiv \alpha_k e^{i\beta k} \tag{III.83.b}$$

According to the notation used by Fry and Kattawar [15], we define the parameters

$$\begin{aligned} \varepsilon &= \beta_1 - \beta_2 \\ \delta &= \beta_3 - \beta_1 \\ \gamma &= \beta_2 - \beta_4 \\ \sigma &= \beta_4 - \beta_1 \\ \lambda &= \beta_2 - \beta_3 \\ \eta &= \beta_4 - \beta_3 \end{aligned} \tag{III.84}$$

With this notation, and taking into account (II.75), we can write the elements of the Mueller matrix corresponding to the same optical medium than **J** as follows

$$\begin{aligned} m_{00} &= \frac{1}{2}\left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2\right) \\ m_{01} &= \frac{1}{2}\left(\alpha_1^2 - \alpha_2^2 - \alpha_3^2 + \alpha_4^2\right) \\ m_{02} &= \alpha_1\alpha_3 \cos\delta + \alpha_2\alpha_4 \cos\gamma \\ m_{03} &= -\alpha_1\alpha_3 \sin\delta - \alpha_2\alpha_4 \sin\gamma \\ m_{10} &= \frac{1}{2}\left(\alpha_1^2 - \alpha_2^2 + \alpha_3^2 - \alpha_4^2\right) \\ m_{11} &= \frac{1}{2}\left(\alpha_1^2 + \alpha_2^2 - \alpha_3^2 - \alpha_4^2\right) \end{aligned} \tag{III.85}$$

$$m_{12} = \alpha_1\alpha_3 \cos\delta - \alpha_2\alpha_4 \cos\gamma$$
$$m_{13} = -\alpha_1\alpha_3 \sin\delta + \alpha_2\alpha_4 \sin\gamma$$
$$m_{20} = \alpha_1\alpha_4 \cos\sigma + \alpha_2\alpha_3 \cos\lambda$$
$$m_{21} = \alpha_1\alpha_4 \cos\sigma + \alpha_2\alpha_3 \cos\lambda$$
$$m_{22} = \alpha_1\alpha_2 \cos\varepsilon + \alpha_3\alpha_4 \cos\eta$$
$$m_{23} = -\alpha_1\alpha_2 \sin\varepsilon + \alpha_3\alpha_4 \sin\eta$$
$$m_{30} = \alpha_1\alpha_4 \sin\sigma + \alpha_2\alpha_3 \sin\lambda$$
$$m_{31} = \alpha_1\alpha_4 \sin\sigma - \alpha_2\alpha_3 \sin\lambda$$
$$m_{32} = \alpha_1\alpha_2 \sin\varepsilon - \alpha_3\alpha_4 \sin\eta$$
$$m_{33} = \alpha_1\alpha_2 \cos\varepsilon - \alpha_3\alpha_4 \cos\eta$$

With the expressions (III.85) for the elements m_{ij} the fulfillment of the nine following equalities can be proved

$$\left(m_{00}+m_{11}\right)^2-\left(m_{01}+m_{10}\right)^2=\left(m_{22}+m_{33}\right)^2+\left(m_{32}-m_{23}\right)^2=4\alpha_1^2\alpha_2^2 \qquad \text{(III.86.a)}$$

$$\left(m_{00}-m_{11}\right)^2-\left(m_{01}-m_{10}\right)^2=\left(m_{22}-m_{33}\right)^2+\left(m_{32}+m_{23}\right)^2=4\alpha_3^2\alpha_4^2 \qquad \text{(III.83.b)}$$

$$\left(m_{00}+m_{10}\right)^2-\left(m_{01}+m_{11}\right)^2=\left(m_{02}+m_{12}\right)^2+\left(m_{03}+m_{13}\right)^2=4\alpha_2^2\alpha_3^2 \qquad \text{(III.83.c)}$$

$$\left(m_{00}-m_{10}\right)^2-\left(m_{01}-m_{11}\right)^2=\left(m_{02}-m_{12}\right)^2+\left(m_{03}-m_{13}\right)^2=4\alpha_1^2\alpha_4^2 \qquad \text{(III.83.d)}$$

$$\left(m_{00}+m_{01}\right)^2-\left(m_{10}+m_{11}\right)^2=\left(m_{20}+m_{21}\right)^2+\left(m_{30}+m_{31}\right)^2=4\alpha_2^2\alpha_4^2 \qquad \text{(III.83.e)}$$

$$\left(m_{00}-m_{01}\right)^2-\left(m_{10}-m_{11}\right)^2=\left(m_{20}-m_{21}\right)^2+\left(m_{30}-m_{31}\right)^2=4\alpha_1^2\alpha_3^2 \qquad \text{(III.83.f)}$$

$$m_{02}m_{03}-m_{12}m_{13}=m_{22}m_{23}+m_{32}m_{33}=-2\alpha_1\alpha_2\alpha_3\alpha_4 \sin\left(\beta_1-\beta_2+\beta_3-\beta_4\right) \qquad \text{(III.87.a)}$$

$$m_{02}m_{12}-m_{02}m_{13}=m_{31}m_{20}-m_{30}m_{21}=-2\alpha_1\alpha_2\alpha_3\alpha_4 \sin\left(\beta_1+\beta_2-\beta_3-\beta_4\right) \qquad \text{(III.87.b)}$$

$$m_{20}m_{30}-m_{21}m_{31}=m_{22}m_{32}+m_{23}m_{33}=2\alpha_1\alpha_2\alpha_3\alpha_4 \sin\left(\beta_1-\beta_2-\beta_3+\beta_4\right) \qquad \text{(III.87.c)}$$

The equalities (III.87) can be replaced by the following three

$$m_{22}^2-m_{23}^2+m_{32}^2-m_{33}^2=m_{02}^2-m_{03}^2-m_{12}^2+m_{13}^2=4\alpha_1\alpha_2\alpha_3\alpha_4 \cos\left(\beta_1-\beta_2+\beta_3-\beta_4\right)$$
$$m_{22}^2-m_{32}^2+m_{23}^2-m_{33}^2=m_{29}^2-m_{30}^2-m_{21}^2+m_{31}^2=4\alpha_1\alpha_2\alpha_3\alpha_4 \cos\left(\beta_1-\beta_2-\beta_3+\beta_4\right) \qquad \text{(III.88)}$$
$$m_{20}^2-m_{21}^2+m_{30}^2-m_{31}^2=m_{03}^2-m_{13}^2+m_{02}^2-m_{12}^2=4\alpha_1\alpha_2\alpha_3\alpha_4 \cos\left(\beta_1+\beta_2-\beta_3-\beta_4\right)$$

The relations (III.86) have been obtained by Abhyankar and Fymat [13], by completing the system of nine equalities with three quartic relations. Later, Fry and Kattawar [15] have shown that these quartic relations can be replaced by the (III.87) or (III.88), which are simpler. The system of nine independent equalities formed by the (III.86) with (III.87) or (III.88) is, as we have shown, equivalent to the other systems of nine restrictive independent equalities studied before.

From the relations (III.57) and (III.74) it is easy to obtain the following two interesting equalities

$$m_{01}^2 + m_{02}^2 + m_{03}^2 = m_{10}^2 + m_{20}^2 + m_{30}^2 \tag{III.89}$$

$$\sum_{i,j=0}^{3} m_{ij}^2 = 4m_{00}^2 \tag{III.90}$$

The equality (III.89) was yet obtained in (III.27.b) from an explicit form of a generic N-type Mueller matrix as a function of the equivalent parameters, according with the theorems EGT and PDT. Besides, the equality (III.90), which is also obtained by adding the relations (III.86), expresses a property that, as we will see, is very useful in order to distinguish the N-type optical media from those that depolarize the light.

Let us consider now the G-type Mueller matrix **M** as the sum of a certain number of N-type Mueller matrices. Taking into account this fact, it can be shown that the following inequalities are fulfilled [15]

$$(m_{00} + m_{11})^2 - (m_{01} + m_{10})^2 \geq (m_{22} + m_{33})^2 + (m_{32} - m_{23})^2 \tag{III.91.a}$$

$$(m_{00} - m_{11})^2 - (m_{01} - m_{10})^2 \geq (m_{22} - m_{33})^2 + (m_{32} + m_{23})^2 \tag{III.91.b}$$

$$(m_{00} + m_{10})^2 - (m_{01} + m_{11})^2 \geq (m_{02} + m_{12})^2 + (m_{03} + m_{13})^2 \tag{III.91.c}$$

$$(m_{00} - m_{10})^2 - (m_{01} - m_{11})^2 \geq (m_{02} - m_{12})^2 + (m_{03} - m_{13})^2 \tag{III.91.d}$$

$$(m_{00} + m_{01})^2 - (m_{10} + m_{11})^2 \geq (m_{20} + m_{21})^2 + (m_{30} + m_{31})^2 \tag{III.91.e}$$

$$(m_{00} - m_{01})^2 - (m_{10} - m_{11})^2 \geq (m_{20} - m_{21})^2 + (m_{30} - m_{31})^2 \tag{III.91.f}$$

To the equality (III.90) corresponds now the inequality

$$\sum_{i,j=0}^{3} m_{ij}^2 \leq 4m_{00}^2 \tag{III.92}$$

To finish this section, we will obtain a set of inequalities that are fulfilled for any Mueller matrix.

The elements of an N-type Mueller matrix **M** can be written by means of the notation given in (II.75). By applying the inequality

$$x^2 + y^2 \geq \pm 2xy \tag{III.93}$$

in the expressions (II.75.b) we see that

$$\alpha_i^2 + \alpha_j^2 \geq \pm 2\beta_{ij} \tag{III.94.a}$$

$$\alpha_i^2 + \alpha_j^2 \geq \pm 2\gamma_{ij} \tag{III.94.b}$$

Now, taking into account the expression (II.75.a) it is easy to demonstrate that the following inequalities are fulfilled

$$\begin{aligned}
m_{00} + m_{11} &\geq \pm\left(m_{22} + m_{33}\right) \\
m_{00} - m_{11} &\geq \pm\left(m_{22} - m_{33}\right) \\
m_{00} - m_{11} &\geq \pm\left(m_{32} - m_{23}\right) \\
m_{00} + m_{10} &\geq \pm\left(m_{02} + m_{12}\right) \\
m_{00} + m_{10} &\geq \pm\left(m_{03} + m_{13}\right) \\
m_{00} - m_{10} &\geq \pm\left(m_{02} - m_{12}\right) \\
m_{00} - m_{10} &\geq \pm\left(m_{03} - m_{13}\right) \\
m_{00} + m_{01} &\geq \pm\left(m_{20} + m_{21}\right) \\
m_{00} + m_{01} &\geq \pm\left(m_{30} + m_{31}\right) \\
m_{00} - m_{01} &\geq \pm\left(m_{20} - m_{21}\right) \\
m_{00} - m_{01} &\geq \pm\left(m_{30} - m_{31}\right)
\end{aligned} \tag{III.95.a}$$

$$m_{00} \geq \pm m_{ij}, \quad \forall i, j \tag{III.95.b}$$

These inequalities, which are additive and thus must be fulfilled for any Mueller matrix without exception, have been recently shown by R.W. Schaefer [16], whose argument for the deduction has been used here.

III.5. Restrictive relations in a Matrix V in the CVF formalism

As the Mueller matrices, the N-type **V** matrices depend, in general, on seven independent parameters. This implies that , in addition to the restrictions (II.55), which are inherent to the definition of the matrix **V**, there must be a set of nine restrictions among its elements.

The expression (II.79) shows the form of a matrix **V** as a function of its corresponding Jones matrix **J**. In (II.79) we can see that the product of the extreme elements of a row, column or diagonal, is equal to the product of their corresponding intermediate elements. This fact implies the existence of the ten following equalities [13]

$$\upsilon_{00}\upsilon_{03} = \upsilon_{01}\upsilon_{02} \quad \text{(III.96.a)}$$

$$\upsilon_{00}\upsilon_{30} = \upsilon_{10}\upsilon_{20} \quad \text{(III.96.b)}$$

$$\upsilon_{30}\upsilon_{33} = \upsilon_{31}\upsilon_{32} \quad \text{(III.96.c)}$$

$$\upsilon_{03}\upsilon_{33} = \upsilon_{13}\upsilon_{23} \quad \text{(III.96.d)}$$

$$\upsilon_{00}\upsilon_{33} = \upsilon_{11}\upsilon_{22} \quad \text{(III.96.e)}$$

$$\upsilon_{03}\upsilon_{30} = \upsilon_{12}\upsilon_{21} \quad \text{(III.96.f)}$$

$$\upsilon_{01}\upsilon_{31} = \upsilon_{11}\upsilon_{21} \quad \text{(III.96.g)}$$

$$\upsilon_{10}\upsilon_{13} = \upsilon_{11}\upsilon_{12} \quad \text{(III.96.h)}$$

$$\upsilon_{02}\upsilon_{32} = \upsilon_{12}\upsilon_{22} \quad \text{(III.96.i)}$$

$$\upsilon_{20}\upsilon_{23} = \upsilon_{21}\upsilon_{22} \quad \text{(III.96.j)}$$

Only eight of these equalities are independent. So, for example, the first eight are independent. The ninth independent equality can be either one of the following

$$\upsilon_{01}\upsilon_{32} = \upsilon_{10}\upsilon_{23} \quad \text{(III.97.a)}$$

$$\upsilon_{02}\upsilon_{31} = \upsilon_{20}\upsilon_{13} \quad \text{(III.97.b)}$$

$$\upsilon_{02}\upsilon_{13} = \upsilon_{10}^{*}\upsilon_{23}^{*} \quad \text{(III.97.c)}$$

$$\upsilon_{20}\upsilon_{13} = \upsilon_{01}^{*}\upsilon_{32}^{*} \quad \text{(III.97.d)}$$

The equalities (III.96.a-g) only contain real quantities. We can complete the system of nine equalities by adding the (III.96.h), (III.96.i) and (III.97.a), which, taking into account (II.55), can be reduced to real expressions of the form

$$\upsilon_{01}\upsilon_{31}+\upsilon_{22}\upsilon_{12}=\upsilon_{11}\upsilon_{21}+\upsilon_{02}\upsilon_{32} \tag{III.98.a}$$

$$\upsilon_{10}\upsilon_{13}+\upsilon_{22}\upsilon_{21}=\upsilon_{11}\upsilon_{12}+\upsilon_{20}\upsilon_{23} \tag{III.98.b}$$

$$\upsilon_{01}\upsilon_{32}+\upsilon_{20}\upsilon_{13}=\upsilon_{10}\upsilon_{23}+\upsilon_{02}\upsilon_{31} \tag{III.98.c}$$

Now, we are going to study others systems of nine restrictions in N-type matrices **V** that, although equivalent among them, are presented in a different way and are occasionally useful.

In the CVF formalism, the quadratic form *F* corresponding to a light beam with an associated coherence vector **D**, can be written as

$$\mathbf{F}=2\mathbf{D}^{T}\mathbf{h}\mathbf{D} \tag{III.99}$$

where $\mathbf{D}^{\mathrm{T}}$ is the transposed row vector of the column vector **D**, and **h** is the matrix

$$\mathbf{h}=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix} \tag{III.100}$$

Let us consider a light beam characterized by a coherency matrix **ρ** and a coherency vector **D**, which passes through an N-type optical medium whose associated matrices in the formalisms CMF and CVF are **J** and **V** respectively. The emerging light beam will be characterized by a coherency matrix **ρ’** and a coherency vector **D’** given by

$$\boldsymbol{\rho}'=\mathbf{J}\boldsymbol{\rho}\mathbf{J}^{+} \tag{III.101}$$

$$\mathbf{D}'=\mathbf{V}\mathbf{D} \tag{III.102}$$

Taking into account the expressions (III.64), (III.99), (III.101) and (III.102), we can write

$$F'=2\mathbf{D}'^{T}\mathbf{h}\mathbf{D}'=2\mathbf{D}^{T}\mathbf{V}^{T}\mathbf{h}\mathbf{V}\mathbf{D} \tag{III.103}$$

$$F'=4\det\boldsymbol{\rho}'=4\left|\det\mathbf{J}\right|^{2}\det\boldsymbol{\rho}=4\left|\det\mathbf{J}\right|^{2}\mathbf{D}^{T}\mathbf{h}\mathbf{D} \tag{III.104}$$

so that

$$\mathbf{D}^{T}\mathbf{V}^{T}\mathbf{h}\mathbf{V}\mathbf{D}=\left|\det\mathbf{J}\right|^{2}\mathbf{D}^{T}\mathbf{h}\mathbf{D} \tag{III.105}$$

The equality (III.105) is satisfied for any vector **D**, and thus

$$\mathbf{V}^T\mathbf{h}\mathbf{V} = \left|\det \mathbf{J}\right|^2 \mathbf{h} \qquad \text{(III.106)}$$

By writing (III.106) as a function of the elements v_{ij} of the matrix **V**, and eliminating $|\det\mathbf{J}|^2$, we obtain the following system of nine restrictive equalities

$$\begin{aligned}
\upsilon_{30}\upsilon_{03} + \upsilon_{00}\upsilon_{33} + \upsilon_{31}\upsilon_{02} + \upsilon_{01}\upsilon_{32} &= \upsilon_{20}\upsilon_{13} + \upsilon_{10}\upsilon_{23} + \upsilon_{21}\upsilon_{12} + \upsilon_{11}\upsilon_{22} \\
\upsilon_{30}\upsilon_{01} + \upsilon_{00}\upsilon_{31} &= \upsilon_{20}\upsilon_{11} + \upsilon_{10}\upsilon_{21} \\
\upsilon_{30}\upsilon_{02} + \upsilon_{00}\upsilon_{32} &= \upsilon_{20}\upsilon_{12} + \upsilon_{10}\upsilon_{22} \\
\upsilon_{31}\upsilon_{03} + \upsilon_{01}\upsilon_{33} &= \upsilon_{21}\upsilon_{13} + \upsilon_{11}\upsilon_{23} \\
\upsilon_{32}\upsilon_{03} + \upsilon_{02}\upsilon_{33} &= \upsilon_{22}\upsilon_{13} + \upsilon_{12}\upsilon_{23} \\
\upsilon_{00}\upsilon_{30} &= \upsilon_{20}\upsilon_{10} \\
\upsilon_{01}\upsilon_{31} &= \upsilon_{21}\upsilon_{11} \\
\upsilon_{02}\upsilon_{32} &= \upsilon_{22}\upsilon_{12} \\
\upsilon_{03}\upsilon_{33} &= \upsilon_{23}\upsilon_{13}
\end{aligned} \qquad \text{(III.107)}$$

We know that if a certain N-type matrix **V** corresponds to an N-type Mueller matrix **M**, $\mathbf{V}^+$ corresponds to $\mathbf{M}^T$. Thus, the matrix $\mathbf{V}^+$ represents an N-type optical medium and must satisfy the same restrictions than **V**, given in (III.107). So, the following equality must be fulfilled

$$\mathbf{V}^*\mathbf{h}\mathbf{V}^+ = \left|\det \mathbf{J}^+\right|^2 \mathbf{h} \qquad \text{(III.108)}$$

This equality gives a system of nine restrictive equalities among the elements v_{ij}, which is equivalent to the expressed in (III.107) and it is obtained by transposing the indexes of all the elements.

From (III.106) and (III.108), and taking into account that $|\det\mathbf{J}^T| = |\det\mathbf{J}|$, we deduce the following matricial equality

$$\mathbf{V}^T\mathbf{h}\mathbf{V} = \mathbf{V}^*\mathbf{h}\mathbf{V}^+ = \left|\det \mathbf{J}^+\right|^2 \mathbf{h} \qquad \text{(III.109)}$$

which includes (III.106) and (III.108) as particular cases.

III.6. Norm condition in Mueller matrices

Given any Mueller matrix **M** we can define a positive defined norm Γ_{M} (**M**) as [42]

$$\varGamma_{\mathbf{M}}\left(\mathbf{M}\right)\equiv\left[\mathrm{tr}\left(\mathbf{M}^{T}\mathbf{M}\right)\right]^{1/2}=\left[\mathrm{tr}\left(\mathbf{M}\mathbf{M}^{T}\right)\right]^{1/2}=\left[\sum_{i,j=0}^{3}m_{ij}^{2}\right]^{1/2} \tag{III.110}$$

As we have seen in (III.92), $\varGamma_{\mathrm{M}}$ (**M**) satisfy the condition

$$\varGamma_{\mathbf{M}}^{2}\left(\mathbf{M}\right)\leq 4m_{00}^{2} \tag{III.111}$$

so that the equality

$$\varGamma_{\mathbf{M}}^{2}\left(\mathbf{M}\right)=4m_{00}^{2} \tag{III.112}$$

occurs when a Mueller matrix **M** is associated with an N-type optical medium. It is worth mentioning that the element m_{00} represents the transmittance T_N of the optical medium for incoming non-polarized light.

We have seen that (III.112) is a necessary condition for **M** to be an N-type one. Now we are going to demonstrate that (III.112) is also a sufficient condition.

Let us supposed the condition (III.112) is fulfilled. Then, the six equalities (III.86) must be fulfilled, because if not, at least one of the inequalities (III.91) must be fulfilled and thus we would obtain $\varGamma_{\mathbf{M}}^{2}\left(\mathbf{M}\right)<4m_{00}^{2}$, which is not in agreement with the hypothesis. In order to obtain a system of nine independent restrictions we need three more, as for example the (III.88).

The inequalities (III.63) and (III.78) can be written as

$$x_1\geq r_1 \tag{III.113.a}$$
$$y_1\geq r_1 \tag{III.113.b}$$
$$z_1\geq r_1 \tag{III.113.c}$$
$$x_2\geq r_2 \tag{III.114.a}$$
$$y_2\geq r_2 \tag{III.114.b}$$
$$z_2\geq r_2 \tag{III.114.c}$$

where the following parameters have been defined

$$r_1=m_{10}^2+m_{20}^2+m_{30}^2-m_{00}^2 \tag{III.115.a}$$
$$x_1=m_{01}^2-m_{11}^2-m_{21}^2-m_{31}^2 \tag{III.115.b}$$
$$y_1=m_{02}^2-m_{12}^2-m_{22}^2-m_{32}^2 \tag{III.115.c}$$
$$z_1=m_{03}^2-m_{13}^2-m_{23}^2-m_{33}^2 \tag{III.115.d}$$

$$r_2 = m_{01}^2 + m_{02}^2 + m_{03}^2 - m_{00}^2 \quad \text{(III.116.a)}$$

$$x_2 = m_{10}^2 - m_{11}^2 - m_{12}^2 - m_{13}^2 \quad \text{(III.116.b)}$$

$$y_2 = m_{20}^2 - m_{21}^2 - m_{22}^2 - m_{23}^2 \quad \text{(III.116.c)}$$

$$z_2 = m_{30}^2 - m_{31}^2 - m_{32}^2 - m_{33}^2 \quad \text{(III.116.d)}$$

From the equalities (III.86.c-f) we deduce that (III.113.a) and (III.114.a) are the equalities $x_1 = r_1$ and $x_2 = r_2$ respectively.

The equalities (III.88) can be written as

$$y_1 = z_1 \quad \text{(III.117.a)}$$

$$y_2 = z_2 \quad \text{(III.117.b)}$$

$$y_1 = y_2 \quad \text{(III.117.c)}$$

In order to demonstrate the fulfillment of the equalities (III.117) we will suppose that $\mathit{\Gamma}_{\mathbf{M}}^2(\mathbf{M}) = 4m_{00}^2$ and that some of the equalities is not fulfilled, obtaining as a result an absurd. The fact that some of the equalities (III.117) is not fulfilled implies that the equalities in (III.113.b-c) and (III.114.b-c) cannot be fulfilled simultaneously, or that $r_1 \neq r_2$.

By adding on one hand the (III.113) and on the other hand the (III.114) we obtain

$$4m_{00}^2 + 2(a-b) \geq \sum_{i,j=0}^{3} m_{ij}^2 \quad \text{(III.118)}$$

$$4m_{00}^2 + 2(b-a) \geq \sum_{i,j=0}^{3} m_{ij}^2 \quad \text{(III.119)}$$

where

$$\begin{aligned} a &= m_{01}^2 + m_{02}^2 + m_{03}^2 \\ b &= m_{10}^2 + m_{20}^2 + m_{30}^2 \end{aligned} \quad \text{(III.120)}$$

As we are supposing that $\mathit{\Gamma}_{\mathbf{M}}^2(\mathbf{M}) = 4m_{00}^2$, from (III.118) and (III.119) we deduce that

$$a = b \quad \text{(III.121)}$$

and thus

$$r_1 = r_2 \quad \text{(III.122)}$$

The only remaining possibilities are

$$y_1 \geq r_1, \quad \text{or} \quad z_1 \geq r_1, \quad \text{or} \quad y_2 \geq r_2, \quad \text{or} \quad z_2 \geq r_2 \tag{III.123}$$

In the first two cases, by adding Eq. (III.113) we obtain

$$4m_{00}^2 > \sum_{i,j=0}^{3} m_{ij}^2 \tag{III.124}$$

and in the other two remaining cases, by adding (III.114), we obtain again (III.124), which is an absurd because of the starting hypothesis (III.112). Then, it is demonstrated that if the condition (III.112) is fulfilled, the system of nine independent equalities formed by the (III.86) and (III.87), or any other equivalent system of equalities, must be fulfilled. This means that **M** corresponds to an N-type optical medium.

We can summarize these considerations in the following theorem [42]: "Given a Mueller matrix **M**, the necessary and sufficient condition for **M** to correspond to an N-type optical medium is $\Gamma_{\mathbf{M}}(\mathbf{M}) = 2m_{00}$".

The interest in this theorem is derived from the fact that, given a Mueller matrix, we can know if it corresponds to an N-type optical medium by only attending the condition (III.112) (the norm condition), without the verification of the nine independent equalities.

As we will see, the norm condition is very useful in the theoretical development of our dynamic method for the determination of Mueller matrices.

Given a Mueller matrix **M**, it can be normalized as

$$\bar{\mathbf{M}} = \frac{1}{m_{00}} \mathbf{M} \tag{III.125}$$

The matrix $\bar{\mathbf{M}}$ corresponds to an optical medium with the same properties than **M**, except for the fact that the former presents a unity transmittance for non-polarized light ($\bar{m}_{00}$= 1). When **M** is N-type, the normalization (III.125) gives the equality

$$\Gamma_{\mathbf{M}}(\bar{\mathbf{M}}) = 2 \tag{III.126}$$

because

$$\Gamma_{\mathbf{M}}\left(\bar{\mathbf{M}}\right)\equiv\left[\operatorname{tr}\left(\bar{\mathbf{M}}^{T}\bar{\mathbf{M}}\right)\right]^{1/2}=\frac{1}{m_{00}}\left[\sum_{i,j=0}^{3}m_{ij}^{2}\right]^{1/2}=2 \qquad \text{(III.127)}$$

An interesting consequence of (III.95.b) and (III.112) is that an N-type Mueller matrix must have, at least, four nonzero elements (except for the trivial case of the zero matrix).

III.7. Norm condition in Jones matrices

The purpose of this section is to obtain, similarly to the last section, a relation among the elements of a generic Jones matrix as a function of the transmittance T_N of the medium for natural light. It is worth mentioning that, although the states of partially polarized light are not able to be represented by means of the formalism JCF, the Jones matrices contain information of T_N, because, as we will see, T_N can be obtain as the semi-sum of the transmittances in minimum and maximum intensities for random variations of the incident light vector.

Given a Jones matrix **J**, we always can associate two numbers $\gamma(\mathbf{J})$, $\tau(\mathbf{J})$, where γ and τ are, respectively, the maximum and minimum value of the division

$$\frac{\left|\mathbf{J}\boldsymbol{\varepsilon}\right|}{\left|\boldsymbol{\varepsilon}\right|} \qquad \text{(III.128)}$$

with respect to the random variations of the two components of the Jones vector **ε** [1].

Any unitary Jones matrix **U** leaves invariant the module of the Jones vector **ε**, and thus

$$\gamma\left(\mathbf{U}\right)=\tau\left(\mathbf{U}\right)=1 \qquad \text{(III.129)}$$

According to the theorem EGT, any Jones matrix **J** can be written as

$$\mathbf{J}=\mathbf{U}_{1}\mathbf{J}_{\mathbf{P}}\left(0,p_{1},p_{2}\right)\mathbf{U}_{2} \qquad \text{(III.130)}$$

where$\mathbf{U}_1$, $\mathbf{U}_2$ are unitary matrices, and $\mathbf{J}_{\mathbf{P}}$ is the matrix associated with a partial polarizer whose principal transmittances in amplitude are p_1, p_2. Taking into account the expressions (III.129) and (III.130) we obtain

$$\gamma(\mathbf{J}) = \gamma(\mathbf{J_P}) = p_1 \quad \text{(III.131.a)}$$

$$\tau(\mathbf{J}) = \tau(\mathbf{J_P}) = p_2 \quad \text{(III.131.b)}$$

According to the expressions (III.15) and (III.26), the element m_{00} of a generic N-type Mueller matrix **M** corresponding to a Jones matrix **J** can be written as

$$m_{00} = \frac{1}{2}\left(p_1^2 + p_2^2\right) = \frac{1}{2}\left[\gamma^2(\mathbf{J}) + \tau^2(\mathbf{J})\right] \quad \text{(III.132)}$$

Besides, according to (II.73), m_{00} can be expressed as a function of the elements of the matrix **J** as follows

$$m_{00} = \frac{1}{2}\sum_{i,j=1}^{2}\left|J_{ij}\right|^2 \quad \text{(III.133)}$$

and, consequently

$$\frac{1}{2}\sum_{i,j=1}^{2}\left|J_{ij}\right|^2 = p_1^2 + p_2^2 \quad \text{(III.134)}$$

The expression (III.133) indicates to us that the one half of the sum of the squares of the modules of the elements of a Jones matrix is equal to the transmittance in intensity for natural light, of the considered medium.

We can associate to any Jones matrix **J** a norm $\Gamma_{\mathrm{J}}(\mathbf{J})$, defined as positive, given by [42]

$$\Gamma_{\mathbf{J}}(\mathbf{J}) \equiv \left[\mathrm{tr}\left(\mathbf{J}^{+}\mathbf{J}\right)\right]^{1/2} = \left[\mathrm{tr}\left(\mathbf{J}\mathbf{J}^{+}\right)\right]^{1/2} = \left[\sum_{i,j=1}^{2}\left|J_{ij}\right|^2\right]^{1/2} \quad \text{(III.135)}$$

or, taking into account (III.133)

$$\Gamma_{\mathbf{J}}^2(\mathbf{J}) = 2m_{00} = 2T_N \quad \text{(III.136)}$$

By comparing (III.136) with (III.112) we see that

$$\Gamma_{\mathbf{M}}(\mathbf{M}) = \Gamma_{\mathbf{J}}^2(\mathbf{J}) \quad \text{(III.137)}$$

The expression (III.136) shows that the norm condition (III.112) for a Mueller matrix **M** is a manifestation of the norm definition (III.135) for a Jones matrix **J** associated with the same N-type optical medium than **M**. These matrices **M** and **J** can be normalized as

$$\bar{\mathbf{M}} = \frac{1}{m_{00}}\mathbf{M} \qquad \text{(III.138.a)}$$

$$\bar{\mathbf{J}} = \frac{1}{\sqrt{m_{00}}}\mathbf{J} \qquad \text{(III.138.b)}$$

so that

$$\Gamma_{\mathbf{M}}\left(\bar{\mathbf{M}}\right) = \Gamma_{\mathbf{J}}^{2}\left(\bar{\mathbf{J}}\right) = 2 \qquad \text{(III.139)}$$

III.8. Norm condition in matrices V of the formalism CVF.

Given a matrix **V** of the formalism CVF, we can associate to it the positive definite norm $\Gamma_{\mathbf{V}}(\mathbf{V})$ [42]

$$\Gamma_{\mathbf{V}}\left(\mathbf{V}\right) \equiv \left[\mathrm{tr}\left(\mathbf{V}^{+}\mathbf{V}\right)\right]^{1/2} = \left[\mathrm{tr}\left(\mathbf{V}\mathbf{V}^{+}\right)\right]^{1/2} = \left[\sum_{i,j=0}^{3} v_{ij}^{2}\right]^{1/2} \qquad \text{(III.140)}$$

We saw in (II.78) that an N-type matrix **V** can be written as a function of its corresponding Jones matrix as

$$\mathbf{V} = \mathbf{J} \times \mathbf{J}^{*} \qquad \text{(III.141)}$$

where x indicates the Kronecker product. From (III.140) and (III.141) it is easy to obtain

$$\Gamma_{\mathbf{V}}\left(\mathbf{V}\right) = \mathrm{tr}^{2}\left(\mathbf{J}\mathbf{J}^{+}\right) = \mathrm{tr}^{2}\left(\mathbf{J}^{+}\mathbf{J}\right) \qquad \text{(III.142)}$$

The expression (III.142), with (III.135), (III.136) and (III.137) let us to write

$$\Gamma_{\mathbf{V}}\left(\mathbf{V}\right) = \Gamma_{\mathbf{M}}\left(\mathbf{M}\right) = \Gamma_{\mathbf{J}}^{2}\left(\mathbf{J}\right) = 2m_{00} \qquad \text{(III.143)}$$

where the matrices **V**, **M** and **J** correspond to the same N-type optical medium. If the considered medium is G-type, **M** and **V** are related according to (II.51), and then

$$\mathit{\Gamma}_{\mathrm{V}}(\mathbf{V}) = \left[\mathrm{tr}\left(\mathbf{V}^{+}\mathbf{V}\right)\right]^{1/2} = \left[\mathrm{tr}\left(\mathbf{U}^{-1}\mathbf{M}^{T}\mathbf{U}\mathbf{U}^{-1}\mathbf{M}\mathbf{U}\right)\right]^{1/2} \qquad \text{(III.144)}$$

Taking into account that **U** is unitary and that $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$, the expression (III.144) can be written as

$$\mathit{\Gamma}_{\mathrm{V}}(\mathbf{V}) = \mathit{\Gamma}_{\mathrm{M}}(\mathbf{M}) \qquad \text{(III.145)}$$

Thus, this relation has been established for any optical medium with any characteristics.

Any matrix **V** can be normalized as

$$\bar{\mathbf{V}} = \left(\frac{2}{v_{00} + v_{03} + v_{30} + v_{33}}\right)\mathbf{V} \qquad \text{(III.146)}$$

where

$$\frac{v_{00} + v_{03} + v_{30} + v_{33}}{2} = T_N = m_{00} \qquad \text{(III.147)}$$

In analogy with the case of Mueller matrices, it is straightforward to demonstrate that the necessary and sufficient condition for a matrix **V** to correspond to an N-type optical medium is

$$\mathit{\Gamma}_{\mathrm{V}}(\mathbf{V}) = \left(v_{00} + v_{03} + v_{30} + v_{33}\right) \qquad \text{(III.148)}$$

or

$$\mathit{\Gamma}_{\mathrm{V}}(\bar{\mathbf{V}}) = 2 \qquad \text{(III.149)}$$

III.9. Indices of polarization and depolarization.

Given an optical medium $\mathcal{O}$, this has associated two Mueller matrices **M** and **M’**, which correspond to the two possible directions of the incident light over $\mathcal{O}$. By convention we say that **M** corresponds to $\mathcal{O}$ when light passes through $\mathcal{O}$ in the ‘forward’ direction, and **M’** when light passes through $\mathcal{O}$ in the ‘reverse’ direction.

Let us consider the set of Stokes vectors

$$\mathbf{S}_{p_1}=\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\quad \mathbf{S}_{p_2}=\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\quad \mathbf{S}_{p_3}=\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\quad \mathbf{S}_{n_1}=\begin{pmatrix}1\\-1\\0\\0\end{pmatrix},\quad \mathbf{S}_{n_2}=\begin{pmatrix}1\\0\\-1\\0\end{pmatrix},\quad \mathbf{S}_{n_3}=\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \qquad \text{(III.150)}$$

which we denote in an abbreviated form as $\mathbf{S}_{p_i}$, $\mathbf{S}_{n_i}$ $\quad i=1,2,3$.

The matrices and the vectors are referred to the same Cartesian system of axis XY. The vectors $\mathbf{S}_{p_i}$, $\mathbf{S}_{n_i}$ correspond to beams of totally polarized light, so that their associated quadratic forms are $F_{p_i}=0,\ F_{n_i}=0$ with i = 1, 2, 3.

When the optical medium $\mathcal{O}$ is passed through in the forward direction by a light beam whose associated Stokes vector $\mathbf{S}_{p_i}\left(\mathbf{S}_{n_i}\right)$ is one of the given in (III.150), the Stokes vector $\mathbf{S}'_{p_i}\left(\mathbf{S}'_{n_i}\right)$ of the emerging light beam have associated the corresponding quadratic form

$$F'_{p_i}=\sum_{j=0}^{3}\left(m_{j0}^2+m_{ji}^2+2m_{j0}m_{ji}\right) \quad i=1,2,3 \qquad \text{(III.151.a)}$$

or

$$F'_{n_i}=\sum_{j=0}^{3}\left(m_{j0}^2+m_{ji}^2-2m_{j0}m_{ji}\right) \quad i=1,2,3 \qquad \text{(III.151.b)}$$

The mean values of these quadratic forms for each two Stokes vectors associated with light beams with orthogonal states of polarization $\mathbf{S}_{p_i}$, $\mathbf{S}_{n_i}$ are

$$F'_1=\frac{1}{2}\left(F'_{p_1}+F'_{n_1}\right)=m_{00}^2+m_{01}^2-m_{11}^2-m_{21}^2-m_{31}^2-m_{10}^2-m_{20}^2-m_{30}^2 \qquad \text{(III.152.a)}$$

$$F'_2=\frac{1}{2}\left(F'_{p_2}+F'_{n_2}\right)=m_{00}^2+m_{02}^2-m_{12}^2-m_{22}^2-m_{32}^2-m_{10}^2-m_{20}^2-m_{30}^2 \qquad \text{(III.152.b)}$$

$$F'_1=\frac{1}{2}\left(F'_{p_3}+F'_{n_3}\right)=m_{00}^2+m_{03}^2-m_{13}^2-m_{23}^2-m_{33}^2-m_{10}^2-m_{20}^2-m_{30}^2 \qquad \text{(III.152.c)}$$

The global mean value is

$$F'_D = \frac{1}{3}\sum_{i=1}^{3} F'_i = m_{00}^2 - \left(m_{10}^2 + m_{20}^2 + m_{30}^2\right) + \frac{1}{3}\left(m_{01}^2 + m_{02}^2 + m_{03}^2\right) - \frac{1}{3}\sum_{k,l=1}^{3} m_{kl}^2 \quad \text{(III.153)}$$

The value of F'_D gives us the mean value of the squares of the intensities of the depolarized light emerging from $\mathcal{O}$ in the case of light beams passing through $\mathcal{O}$ in the forward direction with Stokes vectors $\mathbf{S}_{p_i}$, $\mathbf{S}_{n_i}$.

If we analyze in detail the meaning of the quadratic forms (III.152), we see that $F'_{p_i}\left(F'_{n_i}\right) = 0$ if and only if $\mathcal{O}$ does not depolarize the light passing through in the forward direction with Stokes vector $\mathbf{S}_{p_i}\left(\mathbf{S}_{n_i}\right)$. Thus, taking into account the inequality $X^2 + Y^2 \geq \pm 2XY$ we obtain

$$F'_{p_i} \geq 0, \quad F'_{n_i} \geq 0, \quad i = 1,2,3 \quad \text{(III.154)}$$

and, consequently, for light passing through $\mathcal{O}$ in the forward direction we can state that

$F'_1 = 0 \Leftrightarrow$ $\mathcal{O}$ does not depolarize the incoming light linearly polarized along the axes X or Y

$F'_2 = 0 \Leftrightarrow$ $\mathcal{O}$ does not depolarize the incoming light linearly polarized along the axes at an angle of ±45º with the axes X and Y

$F'_3 = 0 \Leftrightarrow$ $\mathcal{O}$ does not depolarize the incoming circular polarized light (dextro or levo).

From (III.152) and (III.154) we obtain

$$F'_i \geq 0, \quad i = 1,2,3 \quad \text{(III.155)}$$

and

$$F'_D \geq 0 \quad \text{(III.156)}$$

Thus

$$F'_D = 0 \Leftrightarrow F'_{p_i} = F'_{n_i} \quad i = 1,2,3 \quad \text{(III.157)}$$

This last result shows to us that $F'_D = 0$ is a necessary and sufficient condition for $\mathcal{O}$ not to depolarize the light with Stokes vectors $\mathbf{S}_{p_i}$, $\mathbf{S}_{n_i}$, associated with light beams that passes through $\mathcal{O}$ in the forward direction. However, there can be light with other characteristics that, passing through in the forward direction, is depolarized.

By making a similar development, but considering the light passing through $\mathcal{O}$ in the reverse direction, that is, $\mathcal{O}$ is characterized by the Mueller matrix **M'**, the Stokes vectors $\mathbf{S}''_{p_i}$, $\mathbf{S}''_{n_i}$ corresponding to the emerging light beams have associated the following respective quadratic forms

$$F''_{p_i} = \sum_{j=0}^{3} \left(m_{0j}^2 + m_{ij}^2 + 2m_{0j}m_{ij} \right) \quad i = 1,2,3 \tag{III.158.a}$$

$$F''_{n_i} = \sum_{j=0}^{3} \left(m_{0j}^2 + m_{ij}^2 - 2m_{0j}m_{ij} \right) \quad i = 1,2,3 \tag{III.158.b}$$

The mean value F''_D of the squares of the intensities of the depolarized light emerging from $\mathcal{O}$ in the case of light passing through $\mathcal{O}$ in the reverse direction with Stokes vectors $\mathbf{S}_{p_i}$, $\mathbf{S}_{n_i}$ is given by

$$F''_D = \frac{1}{6}\sum_{i=1}^{3}\left(F''_{p_i} + F''_{n_i} \right) = m_{00}^2 - \left(m_{01}^2 + m_{02}^2 + m_{03}^2 \right) - \frac{1}{3}\left(m_{10}^2 + m_{20}^2 + m_{30}^2 \right) - \frac{1}{3}\sum_{k,l=1}^{3} m_{kl}^2 \tag{III.159}$$

All the conclusions obtained for F'_{p_i}, F'_{n_i}, F'_D when $\mathcal{O}$ is passed through in the forward direction are also valid for $F''_{p_i}, F''_{n_i}, F''_D$ when $\mathcal{O}$ is passed through in the reverse direction.

The positive semidefinite quadratic form

$$F_D = \frac{1}{2}\left(F'_D + F''_D \right) = m_{00}^2 - \frac{1}{3}\left(\sum_{i,j=0}^{3} m_{ij}^2 - m_{00}^2 \right) = \frac{1}{3}\left(4m_{00}^2 - \sum_{i,j=0}^{3} m_{ij}^2 \right) \tag{III.160}$$

can be considered as a mean value of the square of the intensity of the unpolarized emerging light, in the case of light beams with Stokes vectors $\mathbf{S}_{p_i}$, $\mathbf{S}_{n_i}$ passing through $\mathcal{O}$ in both forward and reverse direction.

By comparing the definitions (III.110) and (III.160) we see that

$$F_D = \frac{1}{3}\left[4m_{00}^2 - \Gamma_{\mathrm{M}}(\mathbf{M})\right] \tag{III.161}$$

According to the theorem of the norm given in section (III.6), we can say that $F_D = 0$ if, and only if, the optical medium $\mathcal{O}$ does not depolarize light of any kind whatever the direction that $\mathcal{O}$ is passed through. This fact leads to an interpretation of the physical meaning of the norm, because

$$\Gamma_{\mathrm{M}}^2(\mathbf{M}) = 4m_{00}^2 - 3F_D \tag{III.162}$$

and this expression shows that $\Gamma_{\mathrm{M}}^2(\mathbf{M})$ is equal to the difference between four times the square of the transmittance of the medium for non-polarized light and three times the mean value of F_D.

We will call *Depolarization Factor* to the quantity F_D ($\mathbf{M}$) = F_D ($\mathbf{M'}$), because it gives us a global information about the depolarization produced by the optical medium $\mathcal{O}$.

Taking into account the expressions (II.27) and (II.28), which relate the degree of polarization G of a light beam with its corresponding quadratic form F, given by (II.26), we can define the *Depolarization Index* that corresponds to the optical medium $\mathcal{O}$ as follows

$$G_D = \left(\frac{\sum_{i,j=0}^{3} m_{ij}^2 - m_{00}^2}{3m_{00}^2} \right)^{1/2} = \left(1 - \frac{F_D}{m_{00}^2}\right)^{1/2} \tag{III.163}$$

The ranges of possible values for F_D and G_D are the following

$$0 \le F_D \le m_{00}^2 \tag{III.164}$$

$$0 \le G_D \le 1 \tag{III.165}$$

where the values $F_D = 0,\ G_D = 1$ correspond to N-type media, and $F_D = m_{00}^2,\ G_D = 0$ correspond to an ideal depolarizer, that is, an optical medium that totally depolarizes any light beam passing through it in any direction.

The Mueller matrix corresponding to this medium is

$$\mathbf{M} = \mathbf{M}' = \begin{pmatrix} m_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{III.166}$$

Let us consider now that a unpolarized light beam with intensity equal to the unity is passing through an optical medium $\mathcal{O}$ in the forward direction. The Stokes vector **S'** corresponding to the emerging beam is given by

$$\mathbf{S}' = \mathbf{M}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m_{00} \\ m_{10} \\ m_{20} \\ m_{30} \end{pmatrix} \tag{III.167}$$

We define the *Forward Factor of Polarization* of the optical medium $\mathcal{O}$ as the quadratic form F_P' associated with the Stokes vector **S'** according with the definition (III.26). That is to say

$$F_P' = m_{00}^2 - m_{10}^2 - m_{20}^2 - m_{30}^2 \tag{III.168}$$

Similarly we define the *Reciprocal Factor of Polarization* of the optical medium $\mathcal{O}$ as the quadratic form corresponding to the vector

$$\mathbf{S}'' = \mathbf{M}'\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m_{00} \\ m_{01} \\ m_{02} \\ -m_{03} \end{pmatrix} \tag{III.169}$$

which is

$$F_P'' = m_{00}^2 - m_{01}^2 - m_{02}^2 - m_{03}^2 \tag{III.170}$$

From F_P' and F_P'', we can define the ¨respective *Forward Index of Polarization* G_P' and the *Reciprocal Index of Polarization* G_P'' as

$$G_P' = \left(\frac{m_{10}^2 + m_{20}^2 + m_{30}^2}{m_{00}^2} \right)^{1/2} = \left(1 - \frac{F_P'}{m_{00}^2} \right)^{1/2} \tag{III.171}$$

$$G''_P = \left(\frac{m_{01}^2 + m_{02}^2 + m_{03}^2}{m_{00}^2} \right)^{1/2} = \left(1 - \frac{F''_P}{m_{00}^2} \right)^{1/2} \qquad \text{(III.172)}$$

The parameters F'_P and F''_P or G'_P and G''_P give us information about the capacity of the optical medium $\mathcal{O}$ to polarize non-polarized light. The ranges of possible values of these parameters are

$$0 \le F'_P \le m_{00}^2 \qquad \text{(III.173.a)}$$

$$0 \le F''_P \le m_{00}^2 \qquad \text{(III.173.b)}$$

$$0 \le G'_P \le 1 \qquad \text{(III.174.a)}$$

$$0 \le G''_P \le 1 \qquad \text{(III.174.b)}$$

If $\mathcal{O}$ is N-type, the relation (III.27.b) is fulfilled, which taking with (III.168) and (III.170) gives us the equality

$$F'_P = F''_P \qquad \text{(III.175)}$$

From (II.82) and (III.107) it is easy to see that in the formalism CVF the expressions corresponding to F_D, F'_P and F''_P are the following

$$F_D = \frac{1}{3}\left[\left(v_{00} + v_{03} + v_{30} + v_{33} \right)^2 - \Gamma_V^2 \left(V \right) \right] \qquad \text{(III.176)}$$

$$F'_P = 4\left(v_{00}v_{33} + v_{03}v_{30} - v_{10}v_{23} - v_{20}v_{13} \right) \qquad \text{(III.177.a)}$$

$$F''_P = 4\left(v_{00}v_{33} + v_{03}v_{30} - v_{01}v_{32} - v_{02}v_{31} \right) \qquad \text{(III.177.b)}$$

Moreover, if $\mathcal{O}$ is N-type, the corresponding expressions in the formalism JCF are

$$F_D = 0 \qquad \text{(III.178)}$$

$$F'_P = F''_P = \left| J_1 \right|^2 + \left| J_2 \right|^2 + \left| J_3 \right|^2 + \left| J_4 \right|^2 - 2\,\mathrm{Re}\left(J_1 J_2 J_3^* J_4^* \right) \qquad \text{(III.179)}$$

Chapter IV

Method for the dynamic determination of Mueller matrices

In this chapter we expose the theoretical basis of our dynamic method for the determination of Mueller matrices. The proposed measurement device is basically composed of two linear retarders placed between two linear polarizers. In the space between the retarders is placed the optical medium whose associated Mueller matrix **M** is to be measured. A collimated beam of quasi-monochromatic light arrives to the optical medium after passing through the first polarizer and the first retarder. The emerging beam from this medium passes through the second retarder and the second polarizer, in this order. The intensity I of the final emerging light beam depends on the matrix **M** and the orientations of the polarizers and retarders respect to a reference axis.

In order to perform an automatic operation of the device, the retarders rotate in planes perpendicular to the propagation direction of the light beam passing through them, with a fixed relation between the respective constant angular velocities. The intensity I varies periodically, and by means of a detector and a recorder is obtained the record of the signal corresponding to the considered optical medium. The sixteen elements of the Mueller matrix **M** can be obtained from the Fourier analysis of this signal. The Fourier analysis can be made with the help of a computer.

The experimental device for the measurement can be designed for the study of transmission, reflection, diffusion or diffraction phenomena.

If the rotating retarders used in the measurement device are achromatic, it is possible to make a spectroscopy of Mueller matrices by varying the wavelength λ of the light beam and obtaining the corresponding Mueller matrix. Otherwise, if we use a laser beam we can make a local study in different spatial zones of the sample.

IV.1. Measurement device

The figure IV.1 shows schematically the device for the measurement of Mueller matrices, which is mainly composed of two total linear polarizers $P_1(\theta_1)$, $P_2(\theta_2)$; two non-ideal linear retarders L_1, L_2 and an optical medium $\mathcal{O}$ whose associated Mueller matrix **M** is to be measured. "A" represents a source of quasi-monochromatic light, "DT" a detector of the intensity of light, and "RG" a recorder for the signals detected by DT.

The orientations of the principal axes of the optical media are referred to the positive direction of the X axis of the reference system of coordinates XYZ shown in the figure.

The emerging light beam after passing through the system P_1 L_1 $\mathcal{O}$ L_2 P_2 is characterized by the Stokes vector

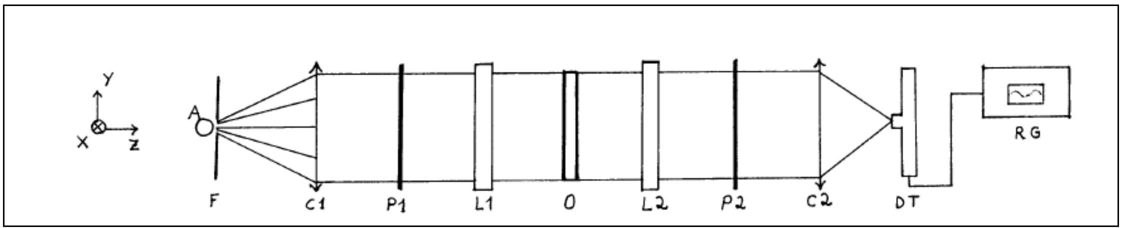
$$\mathbf{S}'' = \frac{1}{16k_a k_b} \mathbf{M_P}(\theta_2) \mathbf{M_L}(\beta_2, \delta_2, k_2) \mathbf{M} \mathbf{M_L}(\beta_1, \delta_1, k_1) \mathbf{M_P}(\theta_1) \mathbf{S} \qquad \text{(IV.1.a)}$$

where

$$k_1 = k'_a / k_a \,, \quad k_2 = k'_b / k_b \qquad \text{(IV.1.b)}$$

and **S** is the Stokes vector associated with the beam of natural light emitted by the source A, so that

$$\mathbf{S} = \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad)$$



(Fig. IV.1) Arrangement of the dynamic device for the determination of Mueller matrices.

The measurement device can be considered as divided into two parts. The first one is composed of P_1, L_1 and $\mathcal{O}$, where the states of polarization of the emerging light beam are generated, which depend on the values of θ_1, β_1, δ_1, k_1 and the elements m_{ij} of the Mueller matrix **M**. This state of polarization is given by the Stokes vector

$$\mathbf{S}' = \frac{1}{4k_a}\mathbf{M}\mathbf{M_L}\left(\beta_1, \delta_1, k_1\right)\mathbf{M_P}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \tag{IV.2}$$

whose elements result in

$$S_i' = \frac{I}{4k_a}\left\{\begin{matrix} \left[1+k_1+(1-k_1)(c_1'c_1+s_1's_1)\right]m_{i0}+ \\ +\left[(1-k_1)c_1'+\left(1+k_1\right)c_1'(c_1'c_1+s_1's_1)+2r_1\cos\delta_1 s_1'(s_1'c_1-c_1's_1\right]m_{i1}+ \\ +\left[(1-k_1)s_1'+\left(1+k_1\right)s_1'(c_1'c_1+s_1's_1)-2r_1\cos\delta_1 c_1'(s_1'c_1-c_1's_1\right]m_{i2}+ \\ +\left[2r_1\sin\delta_1(s_1'c_1-c_1's_1\right]m_{i23} \end{matrix}\right\} \tag{IV.3.a}$$

where

$$\begin{aligned} &s_1 = \sin 2\theta_1, \quad c_1 = \cos 2\theta_1, \\ &s_1' = \sin 2\beta_1, \quad c_1' = \cos 2\beta_1, \\ &r_1 = \left(k_a k_a'\right)^{1/2}. \end{aligned} \tag{IV.3.b}$$

The second part is the system composed of L_2 and P_2, and it is used to analyze the state of polarization of the light. The Mueller matrix **B** associated with this system is

$$\mathbf{B} = \frac{1}{2k_b}\mathbf{M_P}\left(\theta_2\right)\mathbf{M_L}\left(\beta_2, \delta_2, k_2\right) \tag{IV.4}$$

Now, we can write (IV.1) as follows

$$\mathbf{S}'' = \mathbf{B}\mathbf{S}' \tag{IV.5}$$

The detector DT is sensitive to the intensity of light u_0 falling on it

$$u_0 = s_0'' = b_{00}s_0' + b_{01}s_1' + b_{02}s_2' + b_{03}s_3' \tag{IV.6}$$

where the elements b_{0i} (i = 0, 1, 2, 3) of the matrix **B**, obtained from (IV.4), are

$$\begin{aligned} b_{00} &= \frac{1}{4k_b}\left[1+k_1+(1-k_1)(c_1'c_1+s_1's_1)\right] \\ b_{01} &= \frac{1}{4k_b}\left[(1-k_2)c_2'+\left(1+k_2\right)c_2'(c_2'c_2+s_2's_2)+2r_2\cos\delta_2 s_2'(s_2'c_2-c_2's_2\right] \end{aligned} \tag{IV.7.a}$$

$$b_{02}=\frac{1}{4k_b}\left[(1-k_2)s_2'+\left(1+k_2\right)s_2'(c_2'c_2+s_2's_2)+2r_3\cos\delta_2 c_2'(s_2'c_2-c_2's_2\right]$$

$$b_{03}=\frac{1}{4k_b}\left[2r_2\sin\delta_2(c_2's_2-s_2'c_2)\right]$$

where

$$\begin{aligned} s_2&=\sin 2\theta_2, \quad c_2=\cos 2\theta_2, \\ s_2'&=\sin 2\beta_2, \quad c_2'=\cos 2\beta_2, \quad r_2=(k_b k_b')^{1/2} \end{aligned} \qquad \text{(IV.7.b)}$$

In order to simplify the notation we also define the following parameters

$$a_1=k_a+k_a', \quad b_1=k_a-k_a', \quad a_2=k_b+k_b', \quad b_2=k_b-k_b' \qquad \text{(IV.8)}$$

We will suppose that the retarder L_1 rotates around the axis Z with angular velocity ω_1, so that the state of polarization of the light beam emerging from $\mathcal{O}$, generated by the first part of the device, varies continuously with time. Similarly, we will suppose that the retarder L_2 rotates on the axis Z with angular velocity ω_2, so that the second part of the device analyzes dynamically the state of polarization emerging from $\mathcal{O}$. Thus, we can write

$$\beta_1=\omega_1 t \qquad \text{(IV.9.a)}$$

$$\beta_2=\omega_2 t+\alpha_2 \qquad \text{(IV.9.b)}$$

As seen in (IV.9) we suppose that, at the instant $t=0$, the fast axis of L_1 is aligned with the reference axis X, and that the fast axis of L_2 presents an angle α_2 with the axis X.

As we have mentioned above, our aim is to obtain the elements m_{ij} from the Fourier analysis of the periodic intensity signal of the emerging light $u_0(t)$. To obtain a mathematical expression with the form of a Fourier series with a finite number of harmonic components, it is required to force the following relation between the angular velocities of the rotating retarders

$$\omega_2=R\omega_1 \qquad \text{(IV.10)}$$

where R is a rational number

After making the operation indicated in (IV.6), and taking into account (IV.9) and (IV.10), we obtain

$$
\begin{aligned}
u_0 &= h_0 + g_1 \sin 2\beta_1 + h_1 \cos 2\beta_1 + g_2 \sin 4\beta_1 + \\
&+ h_2 \cos 4\beta_1 + g_3 \sin 2R\beta_1 + h_3 \cos 2R\beta_1 + \\
&+ g_4 \sin(2R-2)\beta_1 + h_4 \cos(2R-2)\beta_1 + g_5 \sin(2R+2)\beta_1 + \\
&+ h_5 \cos(2R+2)\beta_1 + g_6 \sin(2R-4)\beta_1 + h_6 \cos(2R-4)\beta_1 + \\
&+ g_7 \sin(2R+4)\beta_1 + h_7 \cos(2R+4)\beta_1 + g_8 \sin 4R\beta_1 + \\
&+ h_8 \cos 4R\beta_1 + g_9 \sin(4R-2)\beta_1 + h_9 \cos(4R-2)\beta_1 + \\
&+ g_{10} \sin(4R+2)\beta_1 + h_{10} \cos(4R+2)\beta_1 + g_{11} \sin(4R-4)\beta_1 + \\
&+ h_{11} \cos(4R-4)\beta_1 + g_{11} \sin(4R+4)\beta_1 + h_{12} \cos(4R+4)\beta_1
\end{aligned}
\tag{IV.11}
$$

where h_i, g_j are coefficients that depend on the elements m_{ij} and also depend on the parameters k_1, k_2, δ_1, δ_2, θ_1, θ_2, α_2.

The expression (IV.11) can be considered as a Fourier series expansion on the multiples of the frequency β_1. We can obtain the Fourier coefficients h_i, g_j by means of the Fourier analysis of the recorded signal u_0. These coefficients depend linearly on the elements, leading to a system of equations in which the unknowns are these elements m_{ij}.

Depending on the value assigned to R we can get up to twelve frequencies that are multiples of β_1, plus the constant term h_0. However, in order to simplify the calculations, it is interesting to obtain a value for R so that the Fourier series development is as short as possible, and generates enough number of Fourier coefficients to obtain the elements m_{ij}. The values R = 1, 2 lead to Fourier series expansions with not sufficient number of harmonics. The value R = 3/2 leads to the following Fourier series expansion [43]

$$
16u_0 = A_0 + \sum_{l=1}^{10} \left(B_l \sin l\beta_1 + A_l \cos l\beta_1 \right) \tag{IV.12}
$$

By writing the coefficients A_i, B_j as functions of the sixteen unknowns m_{ij}, we obtain a system of nineteen equations, where only fifteen are linearly independent. If we have some additional (independent) information about the optical medium $\mathcal{O}$ corresponding to **M**, the corresponding relations should be considered in combination with the above-mentioned fifteen linearly independent equations. For example, if we know that $\mathcal{O}$ is N-type, the following equation should be added [42] in order to obtain the system of sixteen independent equations that is necessary to obtain the sixteen unknowns m_{ij}

$$
4m_{00}^2 = \sum_{i,j=0}^{3} m_{ij}^2 \,. \tag{IV.13}
$$

If the value of the transmittance of intensity for non-polarized light T_N is previously known, then we can add the following equation to the system of equations

$$m_{00} = T_N \tag{IV.14}$$

For a total generality of the method for the determination of the matrix **M**, it is more useful a value for R so that we get sixteen linearly independent equations. The values for R that satisfy this condition are

$$R = \frac{2n_1 + 1}{n_2} \tag{IV.15}$$

where n_1, n_2 are natural numbers, and $n_1 \geq 2$.

For the sake of simplicity in the experimental assembly, the angular velocities ω_1, ω_2 should be as similar as possible.

These requirements are optimally fulfilled for the value

$$R = 5/2 \tag{IV.16}$$

which, combined with (IV.7), (IV.3) and (IV.8) leads to

$$u_0 = A_0 + \sum_{l=1}^{14} \left(B_l \sin l\beta_1 + A_l \cos l\beta_1 \right) \tag{IV.17}$$

where the Fourier coefficients are given by

$$\begin{aligned}
A_0 &= a_1 a_2 m_{00} + a_2 t_1 \left(c_1 m_{01} + s_1 m_{02} \right) + a_1 t_2 \left(c_2 m_{10} + s_2 m_{20} \right) + \\
&+ t_1 t_2 \left[c_1 \left(c_2 m_{11} + s_2 m_{21} \right) + s_1 \left(c_2 m_{12} + s_2 m_{22} \right) \right] \\
B_1 &= \frac{1}{2} b_2 d_1 \left[s_{11} m_{01} - c_{11} m_{02} - s_7 \left(m_{11} + m_{22} \right) + c_7 \left(m_{21} - m_{12} \right) \right] - \\
&- \frac{1}{2} r_2 \sin\delta_2 d_1 \left(c_{11} m_{31} + s_{11} m_{32} \right) \\
A_1 &= \frac{1}{2} b_2 d_1 \left[c_{11} m_{01} + s_{11} m_{02} + c_{11} \left(m_{11} + m_{22} \right) + s_7 \left(m_{21} - m_{12} \right) \right] + \\
&+ \frac{1}{2} r_2 \sin\delta_2 d_1 \left(s_{11} m_{31} - c_{11} m_{32} \right)
\end{aligned} \tag{IV.18}$$

$$B_2 = a_2 b_1 \left(s_1 m_{00} + m_{02} \right) + 2a_2 r_1 \sin \delta_2 c_1 m_{03} +$$
$$+ b_1 t_2 \left[s_1 \left(c_2 m_{10} + s_2 m_{20} \right) + c_2 m_{12} + s_2 m_{22} \right] +$$
$$+ 2r_1 \sin \delta_1 t_2 c_1 \left(c_2 m_{13} + s_2 m_{23} \right)$$

$$A_2 = a_2 b_1 \left(c_1 m_{00} + m_{01} \right) - 2a_2 r_1 \sin \delta_1 s_1 m_{03} +$$
$$+ b_1 t_2 \left[c_1 \left(c_2 m_{10} + s_2 m_{20} \right) + c_2 m_{11} + s_2 m_{21} \right] -$$
$$- 2r_1 \sin \delta_1 t_2 s_1 \left(c_2 m_{13} + s_2 m_{23} \right)$$

$$B_3 = \frac{1}{2} b_2 b_2 \left[s_{11} m_{00} + s_5 m_{01} - c_5 m_{02} - s_7 m_{10} + c_7 m_{20} - s_3 \left(m_{11} + m_{22} \right) + c_3 \left(m_{21} - m_{12} \right) \right] -$$
$$- b_2 r_1 \sin \delta_1 \left(c_{11} m_{03} + c_7 m_{13} + s_7 m_{23} \right) - b_2 r_2 \sin \delta_2 \left(c_{11} m_{30} + c_5 m_{31} + s_5 m_{32} \right) -$$
$$- 2r_1 r_2 \sin \delta_1 \sin \delta_3 s_{11} m_{33}$$

$$A_3 = \frac{1}{2} b_1 b_2 \left[c_{11} m_{00} + c_5 m_{01} + s_5 m_{02} + c_7 m_{10} + s_7 m_{20} + c_3 \left(m_{11} + m_{22} \right) + s_3 \left(m_{21} - m_{12} \right) \right] +$$
$$+ b_2 r_1 \sin \delta_1 \left(s_{11} m_{03} - s_7 m_{13} + c_7 m_{23} \right) + b_1 r_2 \sin \delta_2 \left(s_{11} m_{30} + s_7 m_{31} - c_7 m_{32} \right) -$$
$$- 2r_1 r_2 \sin \delta_1 \sin \delta_2 c_{11} m_{33}$$

$$B_4 = a_2 d_1 \left(s_1 m_{01} + c_1 m_{02} \right) + d_1 t_2 \left[s_1 \left(c_2 m_{11} + s_2 m_{21} \right) + c_1 \left(c_2 m_{12} + s_2 m_{22} \right) \right]$$

$$A_4 = a_2 d_1 \left(c_1 m_{01} - s_1 m_{02} \right) + d_1 t_2 \left[c_1 \left(c_2 m_{11} + s_2 m_{21} \right) - s_1 \left(c_2 m_{12} + s_2 m_{22} \right) \right]$$

$$B_5 = a_1 b_2 \left(s_5 m_{00} - s_3 m_{10} + c_3 m_{20} \right) +$$
$$+ b_2 t_1 \left[s_5 \left(c_1 m_{01} + s_1 m_{02} \right) - s_3 \left(c_1 m_{11} + s_1 m_{12} \right) + c_3 \left(c_1 m_{21} + s_1 m_{22} \right) \right] -$$
$$- 2r_1 \sin \delta_2 c_5 \left[a_1 m_{30} + t_1 \left(c_1 m_{31} + s_1 m_{32} \right) \right]$$

$$A_5 = a_1 b_2 \left(c_5 m_{00} + c_3 m_{10} + s_3 m_{20} \right) +$$
$$+ b_2 t_1 \left[c_5 \left(c_1 m_{01} + s_1 m_{02} \right) + c_3 \left(c_1 m_{11} + s_1 m_{12} \right) + s_3 \left(c_1 m_{21} + s_1 m_{22} \right) \right] +$$
$$+ 2r_2 \sin \delta_2 s_5 \left[a_1 m_{30} + t_1 \left(c_1 m_{31} + s_1 m_{32} \right) \right]$$

$$B_6 = \frac{1}{2} d_1 d_2 \left[s_{13} \left(m_{11} + m_{22} \right) + c_{13} \left(m_{21} - m_{12} \right) \right]$$

$$A_6 = \frac{1}{2} d_1 d_2 \left[c_{13} \left(m_{11} + m_{22} \right) - s_{13} \left(m_{21} - m_{12} \right) \right]$$

$$B_7 = \frac{1}{2} b_2 b_2 \left[s_{10} m_{00} + s_5 m_{01} + c_5 m_{02} + s_8 m_{10} + c_8 m_{20} - s_3 \left(m_{11} - m_{22} \right) + c_3 \left(m_{12} + m_{21} \right) \right] +$$
$$+ b_2 r_1 \sin \delta_1 \left(c_{10} m_{03} + c_8 m_{13} - s_8 m_{23} \right) - b_1 r_2 \sin \delta_2 \left(c_{10} m_{30} + c_5 m_{31} - s_5 m_{32} \right) +$$
$$+ 2 r_1 r_2 \sin \delta_1 \sin \delta_2 s_{10} m_{33}$$

$$A_7 = \frac{1}{2} b_2 b_2 \left[c_{10} m_{00} + c_5 m_{01} - s_5 m_{02} + c_8 m_{10} - s_8 m_{20} + c_3 \left(m_{11} - m_{22} \right) + s_3 \left(m_{12} + m_{21} \right) \right] -$$
$$- b_2 r_1 \sin \delta_1 \left(s_{10} m_{03} + s_8 m_{13} + c_8 m_{23} \right) + b_2 r_2 \sin \delta_2 \left(s_{10} m_{30} + s_5 m_{31} + c_5 m_{32} \right) +$$
$$+ 2 r_1 r_2 \sin \delta_1 \sin \delta_2 c_{10} m_{33}$$

$$B_8 = \frac{1}{2} b_1 d_2 \left[s_{13} m_{10} + c_{13} m_{20} + s_9 \left(m_{11} + m_{22} \right) + c_9 \left(m_{21} - m_{12} \right) \right] +$$
$$+ r_1 \sin \delta_1 d_2 \left(s_{13} m_{23} - c_{13} m_{13} \right)$$

$$A_8 = \frac{1}{2} b_2 d_1 \left[s_{13} m_{10} - c_{13} m_{20} + c_9 \left(m_{11} + m_{22} \right) - s_9 \left(m_{21} - m_{12} \right) \right] +$$
$$+ r_1 \sin \delta_1 d_2 \left(c_{13} m_{23} + s_{13} m_{13} \right)$$

$$B_9 = \frac{1}{2} b_2 d_1 \left[s_{10} m_{01} + c_{10} m_{02} + s_8 \left(m_{11} - m_{22} \right) + c_8 \left(m_{12} + m_{21} \right) \right] +$$
$$+ r_2 \sin \delta_2 d_1 \left(s_{10} m_{32} - c_{10} m_{31} \right)$$

$$A_9 = \frac{1}{2} b_2 d_1 \left[c_{10} m_{01} - s_{10} m_{02} + c_8 \left(m_{11} - m_{22} \right) - s_8 \left(m_{12} + m_{21} \right) \right] +$$
$$+ r_2 \sin \delta_2 d_1 \left(c_{10} m_{32} + s_{10} m_{31} \right)$$

$$B_{10} = a_1 d_2 \left(s_9 m_{10} + c_9 m_{20} \right) + t_1 d_2 \left[c_1 \left(s_9 m_{11} + c_9 m_{21} \right) + s_1 \left(s_9 m_{12} + c_9 m_{22} \right) \right]$$

$$A_{10} = a_1 d_2 \left(c_9 m_{10} - s_9 m_{20} \right) + t_1 d_2 \left[c_1 \left(c_9 m_{11} - s_9 m_{21} \right) + s_1 \left(c_9 m_{12} - s_9 m_{22} \right) \right]$$

$$B_{11} = A_{11} = 0$$

$$B_{12} = \frac{1}{2} b_1 d_2 \left[s_{12} m_{10} + c_{12} m_{20} + s_9 \left(m_{11} - m_{22} \right) + c_9 \left(m_{12} + m_{21} \right) \right] +$$
$$+ r_1 \sin \delta_1 d_2 \left(c_{12} m_{13} - s_{12} m_{23} \right)$$

$$A_{12} = \frac{1}{2} b_1 d_2 \left[c_{12} m_{10} - s_{12} m_{20} + c_9 \left(m_{11} - m_{22} \right) - s_9 \left(m_{12} + m_{21} \right) \right] -$$
$$-r_1 \sin\delta_1 d_2 \left(s_{12} m_{13} + c_{12} m_{23} \right)$$

$$B_{13} = A_{13} = 0$$

$$B_{14} = \frac{1}{2} d_1 d_2 \left[s_{12} \left(m_{11} - m_{22} \right) + c_{12} \left(m_{12} + m_{21} \right) \right]$$

$$A_{14} = \frac{1}{2} d_1 d_2 \left[c_{12} \left(m_{11} - m_{22} \right) - s_{12} \left(m_{12} + m_{21} \right) \right]$$

In (IV.18) and hereafter we use the notation

$$t_1 = \frac{1}{2} a_1 + r_1 \cos\delta_1, \quad t_2 = \frac{1}{2} a_2 + r_2 \cos\delta_2$$

$$d_1 = \frac{1}{2} a_1 - r_1 \cos\delta_1, \quad d_2 = \frac{1}{2} a_2 - r_2 \cos\delta_2$$

$$\tau_1 = \theta_1, \quad \tau_2 = \theta_2, \quad \tau_3 = \alpha_2, \quad \tau_4 = \theta_2 - \theta_1, \quad \tau_5 = \theta_2 - \alpha_2,$$
$$\tau_6 = \theta_1 + \theta_2, \quad \tau_7 = \theta_1 + \alpha_2, \quad \tau_8 = \theta_1 - \alpha_2, \quad \tau_9 = \theta_2 - 2\alpha_2, \qquad \text{(IV.19)}$$
$$\tau_{10} = \theta_1 + \theta_2 - \alpha_2, \quad \tau_{11} = \theta_2 - \theta_1 - \alpha_2, \quad \tau_{12} = \theta_1 + \theta_2 - 2\alpha_2,$$
$$\tau_{13} = \theta_2 - \theta_1 - 2\alpha_2,$$
$$s_i = \sin 2\tau_i, \quad c_i = \cos 2\tau_i, \quad i = 1, 2, ..., 13$$

Now, by cconsidering the elements m_{ij} as the unknowns and the remainder parameters as the known data, the inversion of the system of equations (IV.18), leads to

$$m_{11} = \frac{s_{13} B_6 + c_{13} A_6 + s_{12} B_{14} + c_{12} A_{14}}{d_1 d_2}$$
$$m_{22} = \frac{s_{13} B_6 + c_{13} A_6 - s_{12} B_{14} - c_{12} A_{14}}{d_1 d_2} \qquad \text{(IV.20)}$$
$$m_{12} = \frac{-c_{13} B_6 + s_{13} A_6 + c_{12} B_{14} - s_{12} A_{14}}{d_1 d_2}$$

$$m_{11} = \frac{s_{13}B_6 + c_{13}A_6 + s_{12}B_{14} + c_{12}A_{14}}{d_1 d_2}$$

$$m_{22} = \frac{s_{13}B_6 + c_{13}A_6 - s_{12}B_{14} - c_{12}A_{14}}{d_1 d_2}$$

$$m_{12} = \frac{-c_{13}B_6 + s_{13}A_6 + c_{12}B_{14} - s_{12}A_{14}}{d_1 d_2}$$

$$m_{21} = \frac{c_{13}B_6 - s_{13}A_6 + c_{12}B_{14} - s_{12}A_{14}}{d_1 d_2}$$

$$m_{01} = \frac{s_1 B_4 + c_1 A_4 - d_1 t_2 \left(c_2 m_{11} + s_2 m_{21}\right)}{a_2 d_1}$$

$$m_{02} = \frac{c_1 B_4 - s_1 A_4 - d_1 t_2 \left(c_2 m_{12} + s_2 m_{22}\right)}{a_2 d_1}$$

$$m_{10} = \frac{s_9 B_{10} + c_9 A_{10} - t_1 d_2 \left(c_1 m_{11} + s_1 m_{12}\right)}{a_1 d_2}$$

$$m_{20} = \frac{c_9 B_{10} - s_9 A_{10} - t_1 d_2 \left(c_1 m_{21} + s_1 m_{22}\right)}{a_1 d_2}$$

$$m_{13} = \frac{2\left(c_{12}B_{12} - s_{12}A_{12}\right) - b_1 d_2 \left[m_{20} + s_1\left(m_{22} - m_{11}\right) + c_1\left(m_{12} + m_{21}\right)\right]}{2 r_1 \sin\delta_1 d_2}$$

$$m_{23} = \frac{-2\left(s_{12}B_{12} + c_{12}A_{12}\right) + b_1 d_2 \left[m_{10} + c_1\left(m_{11} - m_{22}\right) + s_1\left(m_{12} + m_{21}\right)\right]}{2 r_1 \sin\delta_1 d_2}$$

$$m_{31} = \frac{2\left(s_{11}A_1 - c_{11}B_1\right) - b_2 d_1 \left[m_{02} + s_2\left(m_{11} + m_{22}\right) + c_2\left(m_{12} - m_{21}\right)\right]}{2 r_2 \sin\delta_2 d_1}$$

$$m_{00} = \frac{A_0 - \left(\frac{1}{2} a_1 a_2 + a_2 r_1 \cos\delta_1 + a_1 r_2 \cos\delta_2\right)\left(c_1 m_{01} + s_1 m_{02} + c_2 m_{10} + s_2 m_{20}\right)}{a_1 a_2} -$$

$$- \frac{t_1 t_2 \left[c_1\left(c_2 m_{11} + s_2 m_{21}\right) + s_1\left(c_2 m_{12} + s_2 m_{22}\right)\right]}{a_1 a_2}$$

$$m_{03} = \frac{c_1 B_2 - s_1 A_2 - a_2 b_1 \left(c_1 m_{02} - s_1 m_{01}\right) + b_1 d_2 \left[s_1\left(c_2 m_{11} + s_2 m_{21}\right) - c_1\left(c_2 m_{12} + s_2 m_{22}\right)\right]}{2 a_2 r_1 \sin\delta_1} -$$

$$- \frac{t_2\left(c_2 m_{13} + s_2 m_{23}\right)}{a_2}$$

IV.2. Calibration

The polarimetric characteristics of the measurement device (absolute Mueller polarimeter) are defined by the parameters δ_1, δ_2, k_1, k_2, θ_1, θ_2, α_2, R. In the last section the possible values for R has been discussed, with a definitive choice for R = 5/2.

The retarders L_1 and L_2 can be chosen so that the nominal value of the retardation is equal to a previously established value, but it should be noted that this nominal value is always subject to a certain margin of error, partly caused by the effects of the multiple internal reflections [37, 44, 45]. Moreover, k_1 and k_2 always differ from the ideal value, i.e. $k_1 = k_2 = 1$ [37].

The angular parameters θ_1, θ_2, α_2 are easily controllable. However, the imposing of previously established values can induce errors in the measurements.

These considerations make advisable a calibration on the device to obtain the effective values of the characteristics parameters of the system.

In this section we expose the calibration procedure, with the advantage of not being necessary an optical medium as test, which would induce additional errors on the determination of the parameters.

If the system does not have any optical medium in the place of the sample, the associated Mueller matrix is the Identity matrix, and then the system of equations (IV.18) becomes

$$A_0 = a_1 a_2 + t_1 t_2 \cos 2(\theta_2 - \theta_1)$$

$$B_1 = -b_2 d_1 \sin 2(\theta_1 + \alpha_2)$$

$$A_1 = b_2 d_1 \cos 2(\theta_1 + \alpha_2)$$

$$B_2 = b_1 \left(a_2 s_1 + t_2 s_2 \right)$$

$$A_2 = b_1 \left(a_2 c_1 + t_2 c_2 \right) \tag{IV.21}$$

$$B_3 = \frac{1}{2}\left(b_1 b_2 - 4 r_1 r_2 \sin\delta_1 \sin\delta_2\right) \sin 2\left(\theta_2 - \theta_1 - \alpha_2\right) - b_1 b_2 s_3$$

$$A_3 = \frac{1}{2}\left(b_1 b_2 - 4 r_1 r_2 \sin\delta_1 \sin\delta_2\right) \cos 2\left(\theta_2 - \theta_1 - \alpha_2\right) - b_1 b_2 s_3$$

$$B_4 = d_1 t_2 \sin 2\left(\theta_1 + \theta_2\right)$$

$$A_4 = d_1 t_2 \cos 2(\theta_1 + \theta_2)$$

$$B_5 = b_2 \left[a_1 \sin 2(\theta_2 - \alpha_2) + t_1 \sin 2(\theta_1 - \alpha_2) \right]$$

$$A_5 = b_2 \left[a_1 \cos 2(\theta_2 - \alpha_2) + t_1 \cos 2(\theta_1 - \alpha_2) \right]$$

$$B_6 = d_1 d_2 \sin 2(\theta_2 - \theta_1 - 2\alpha_2)$$

$$A_6 = d_1 d_2 \cos 2(\theta_2 - \theta_1 - 2\alpha_2)$$

$$B_7 = \left(\frac{1}{2} b_1 b_2 + 2 r_1 r_2 \sin \delta_1 \sin \delta_2 \right) \sin 2(\theta_1 + \theta_2 - \alpha_2)$$

$$A_7 = \left(\frac{1}{2} b_1 b_2 + 2 r_1 r_2 \sin \delta_1 \sin \delta_2 \right) \cos 2(\theta_1 + \theta_2 - \alpha_2)$$

$$B_8 = b_1 d_2 \sin 2(\theta_2 - 2\alpha_2)$$

$$A_8 = b_1 d_2 \cos 2(\theta_2 - 2\alpha_2)$$

$$B_{10} = t_1 d_2 \sin 2(\theta_1 + \theta_2 - 2\alpha_2)$$

$$A_{10} = t_1 d_2 \cos 2(\theta_1 + \theta_2 - 2\alpha_2)$$

$$B_9 = A_9 = B_{12} = A_{12} = B_{14} = A_{14} = 0$$

From these expressions it is easy to prove that

$$\tan 2(\theta_1 + \theta_2) = \frac{B_4}{A_4} \qquad \text{(IV.22.a)}$$

$$\tan 2(\theta_2 - \theta_1 - 2\alpha_2) = \frac{B_6}{A_6} \qquad \text{(IV.22.b)}$$

$$\tan 2(\theta_1 + \theta_2 - 2\alpha_2) = \frac{B_{10}}{A_{10}} \qquad \text{(IV.22.c)}$$

$$\tan 2(\theta_1 + \theta_2 - \alpha_2) = \frac{B_7}{A_7} \qquad \text{(IV.22.d)}$$

$$\tan 2(\theta_1 + \alpha_2) = -\frac{B_1}{A_1} \qquad \text{(IV.22.e)}$$

$$\tan 2(\theta_1 - 2\alpha_2) = \frac{B_8}{A_8} \qquad \text{(IV.22.f)}$$

It is worth mentioning that the values of k_1, k_2 are close to unity in the retarders, and thus, the parameters b_1, b_2 are close to zero. This makes advisable that, when possible, the unknown parameters are extracted from coefficients of Fourier that do not include b_1 or b_2 as global factor. In this sense, to obtain θ_1, θ_2 and α_2 three of the four first relations (IV.22) must be used.

Now, we can consider the angles θ_1, θ_2, α_2 as data, and obtain from them all the angular parameters defined in (IV.19). To simplify later expressions we define

$$D_1 = \frac{1}{2}\frac{\left(\frac{B_{10}s_{13}}{B_6 s_{12}} - 1\right)}{\left(\frac{B_{10}s_{13}}{B_6 s_{12}} + 1\right)}, \quad D_2 = \frac{1}{2}\frac{\left(\frac{B_4 s_{13}}{B_6 s_6} - 1\right)}{\left(\frac{B_4 s_{13}}{B_6 s_6} + 1\right)} \tag{IV.23}$$

From (IV.21) we obtain

$$a_1 a_2 = \frac{s_6 B_4 + c_6 A_4}{\left(\frac{1}{2} - D_1\right)\left(\frac{1}{2} - D_2\right)} = \frac{s_{13} B_6 + c_{13} A_6}{\left(\frac{1}{2} - D_1\right)\left(\frac{1}{2} - D_2\right)} = \frac{s_{12} B_{10} + c_{12} A_{10}}{\left(\frac{1}{2} - D_1\right)\left(\frac{1}{2} - D_2\right)} = \\ = \frac{A_0}{1 + \left(\frac{1}{2} + D_1\right)\left(\frac{1}{2} + D_2\right)c_4} \tag{IV.24.a}$$

$$b_1 b_2 = \frac{s_{11} A_3 - c_{11} B_3}{s_4} \tag{IV.24.b}$$

$$a_1 b_2 = \frac{c_8 B_5 - s_8 A_5}{s_4} \tag{IV.24.c}$$

$$a_2 b_1 = \frac{s_2 A_2 - c_2 B_2}{s_4} \tag{IV.24.d}$$

We also define the following parameters

$$x_1 \equiv \left(a_1 a_2 + a_1 b_2 + a_2 b_1 + b_1 b_2\right) = 4 k_a k_b \tag{IV.25.a}$$

$$x_2 \equiv \left(a_1 a_2 - a_1 b_2 - a_2 b_1 + b_1 b_2\right) = 4 k_a' k_b' \tag{IV.25.b}$$

$$x_3 \equiv \left(a_1 a_2 - a_1 b_2 + a_2 b_1 - b_1 b_2\right) = 4 k_a k_b' \tag{IV.25.c}$$

$$x_4 \equiv \left(a_1 a_2 + a_1 b_2 - a_2 b_1 - b_1 b_2\right) = 4 k_a' k_b \tag{IV.25.d}$$

The parameters of the instrument k_1, k_2, δ_1, δ_2 can be calculated as follows

$$k_1 = \frac{x_4}{x_1} = \frac{x_2}{x_3} \qquad \text{(IV.26.a)}$$

$$k_2 = \frac{x_2}{x_4} = \frac{x_3}{x_1} \qquad \text{(IV.26.b)}$$

$$\cos\delta_1 = D_1 \frac{1+k_1}{k_1^{1/2}} \qquad \text{(IV.27.a)}$$

$$\cos\delta_2 = D_2 \frac{1+k_2}{k_2^{1/2}} \qquad \text{(IV.27.b)}$$

For δ_1, δ_2 we can fix the following range of variation

$$0 \le \delta_i \le \pi, \quad i = 1,2 \qquad \text{(IV.28)}$$

this is convenient because the values $\pi \le \delta_i < 2\pi, \quad i = 1,2$ are equivalent to the values indicated in (IV.28) but with a rotation of $\pi/2$ in the optical axes of the retarder. So, δ_1, δ_2 are determined by (IV.27), not being necessary to know the signs of sin δ_1, sin δ_2, because these are positive.

To avoid indeterminations in the systems (IV.20) and (IV.24), some conditions in the characteristic parameters of the device must be fulfilled. These conditions are

$$\theta_1 \ne \theta_2 \qquad \text{(IV.29)}$$

$$\delta_i \ne 0, \pi \quad i = 1,2 \qquad \text{(IV.30)}$$

The ranges of the acceptable values for the parameters can be summarized in

$$0 < \delta_1, \quad \delta_2 < \pi \qquad \text{(IV.31)}$$

$$-\frac{\pi}{2} < \theta_1, \theta_2, \alpha_2 \le \frac{\pi}{2} \qquad \text{(IV.32)}$$

together with the condition (IV.29)

We get the self-calibration of the device by means of a Fourier analysis of the signal of the light intensity corresponding to the case where there is not any optical medium as sample in the polarimeter. The Fourier coefficients of this analysis are given in (IV.21), and from them the values of the parameters δ_1, δ_2, k_1, k_2, θ_1, θ_2, α_2 can be calculated by means of the relations (IV.22.a.d), (IV.26) and (IV.27). Once these parameters are measured, we can make the measurements of the Mueller matrices, whose elements are obtained with (IV.20), where the Fourier coefficients A_i, B_j correspond to the Fourier analysis of the signal recorded in each measurement.

Sometimes, a tuning of the values δ_1 and δ_2 of the retarders can be convenient. This can be made by means of the use of two Soleil compensators as the retarders L_1 and L_2. Other alternative option, which presents some advantages, is the using of respective sets, each one composed of three commercial retardation sheets, so that the two extreme sheets are equal and with their fast axes aligned. Each of these sets can be called as L(0, δ) L(α, δ') L(0,δ), and according to the theorem T14 is equivalent to a lineal retarder L (θ, Δ) so that [39]

$$\tan 2\theta = \frac{\sin 2\alpha}{\sin\delta \cot\left(\delta'/2\right) + \cos\delta \cos 2\alpha} \tag{IV.33}$$

$$\cos 2\theta = \cos\delta \cos\left(\delta'/2\right) - \sin\delta \sin\left(\delta'/2\right)\cos 2\alpha \tag{IV.34}$$

According to these expressions we see that by means of the tuning of the orientation α of the intermediate retarder we get different equivalent linear retarders with values for θ, Δ in the following ranges

$$|\theta| \le \frac{1}{2}\arctan\frac{\sin\left(\delta'/2\right)}{\left[\sin^2\delta - \sin^2\left(\delta'/2\right)\right]} \tag{IV.35}$$

$$|2\delta - \delta'| \le \Delta \le 2\delta + \delta' \tag{IV.36}$$

Another possibility is the using of two equal sets of two linear retardation sheets. One of these sets are called as L(α, δ) L (0, δ'), and according to the theorem T4 is equivalent to a system L(θ, Δ) R(γ) composed of a linear retarder and a rotator so that [39]

$$\tan\gamma = \frac{\sin 2\alpha}{\cos 2\alpha - \cot\left(\delta/2\right)\cot\left(\delta'/2\right)} \tag{IV.37}$$

$$\tan\left(2\theta - \gamma\right) = \frac{\sin 2\alpha}{\cos 2\alpha + \cot\left(\delta/2\right)\cot\left(\delta'/2\right)} \tag{IV.38}$$

$$\cos^2\left(\Delta/2\right) = \cos^2\left(\frac{\delta+\delta'}{2}\right)\cos^2\alpha + \cos^2\left(\frac{\delta-\delta'}{2}\right)\sin^2\alpha \tag{IV.39}$$

If we use the system L(α, δ) L(0, δ') as the retarder L_1, and the system L(0, δ') L(α, δ) as L_2, the effect of the equivalent rotator of a system is compensated by the other one, because both of the rotators introduce an equal rotation but in the opposite sense. The parameter γ of the equivalent rotator of each system does not depend on its absolute orientation, so the effect of the rotators is compensated even when the two systems of two retarders are rotating.

IV.3 Apparatus signal

In the last section we have seen that the characteristic parameters of the instrument are obtained from a record without any optical medium in the assembly. Hereafter, we will call apparatus signal to any signal obtained on a record of this kind. The parameters obtained from this signal can be used to generate, with the help of a computer, the graphic of an ideal signal corresponding to these parameters, so that it is obtained from (IV.21). In order to get a visual qualitative idea of the accuracy of the measurement, the graphic of the ideal apparatus signal obtained can be compared with the one obtained in the experimental record. The more similar the two signals, the more accurate will be the device*. Examples of ideal signal corresponding to several values of the characteristic parameters of the device are shown in Fig. IV.2-IV.5.

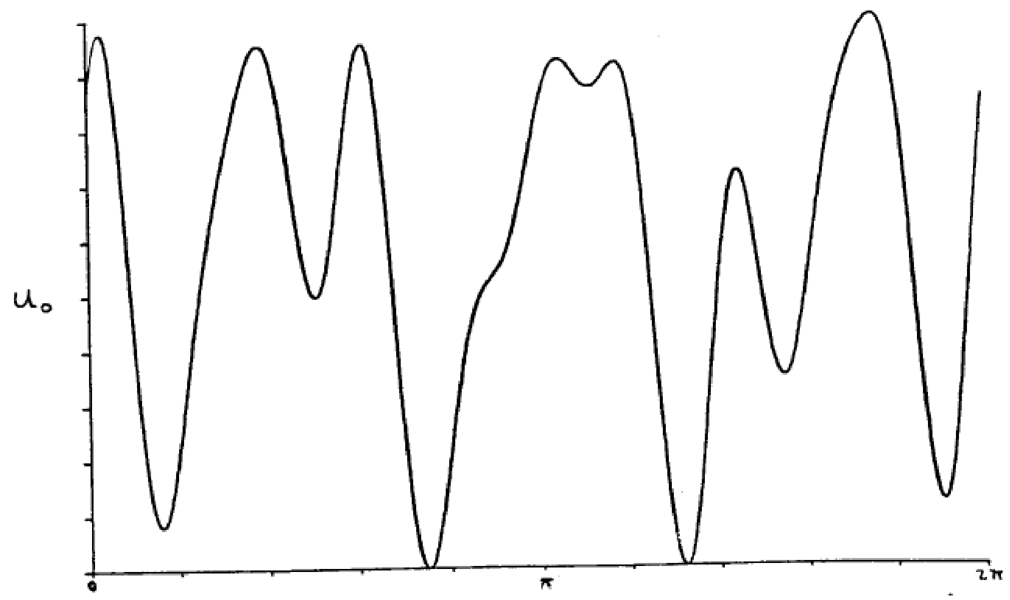


Fig. IV.2. Apparatus signal corresponding to the following values of the parameters of the Mueller polarimeter: $\alpha_2 = 0°$, $\theta_1 = 0°$, $\theta_2 = 22.5°$, $\delta_1 = \delta_2 = 90°$, $g_1 = g_2 = 1$.

**In chapter VI the principal effects that can induce divergences between the ideal apparatus signal and the experimental one, are analyzed and discussed.*

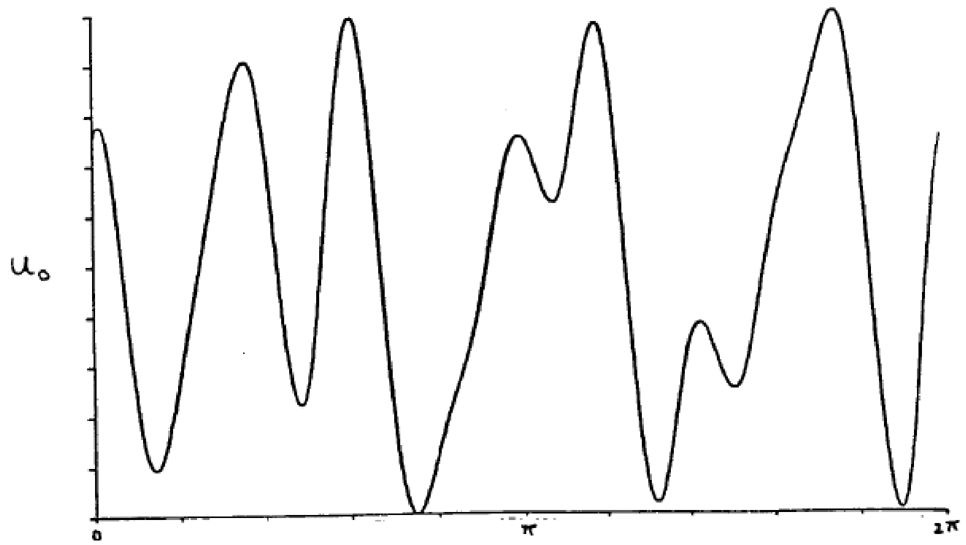


Fig. IV.3 Apparatus signal corresponding to the following values of the parameters of the Mueller polarimeter: $\alpha_2 = 22.5°$, $\theta_1 = 0°$, $\theta_2 = 45°$, $\delta_1 = \delta_2 = 90°$, $g_1 = g_2 = 0.980$.

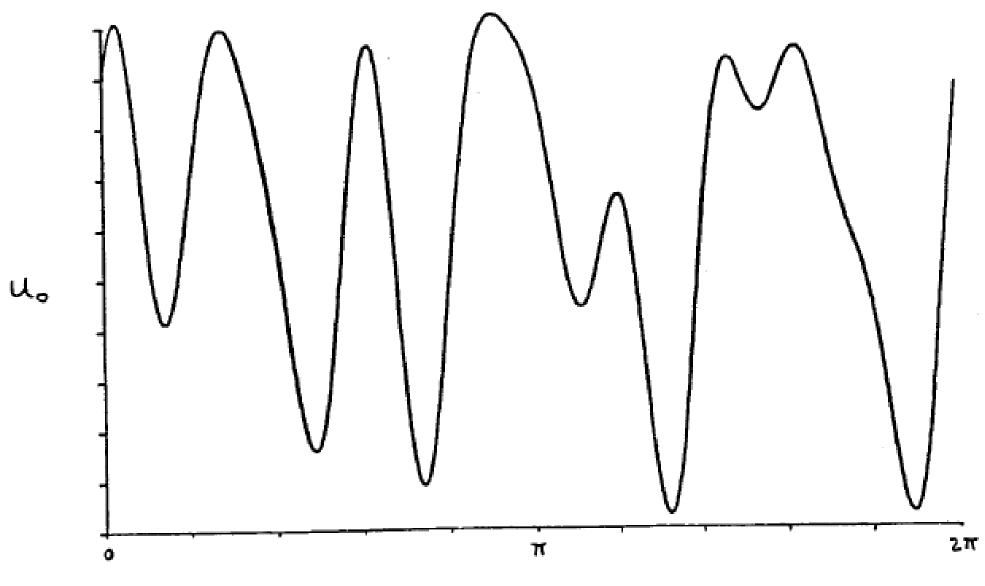


Fig. IV.4 Apparatus signal corresponding to the following values of the parameters of the Mueller polarimeter: $\alpha_2 = 0°$, $\theta_1 = 22.5°$, $\theta_2 = 0°$, $\delta_1 = \delta_2 = 90°$, $g_1 = g_2 = 1$.

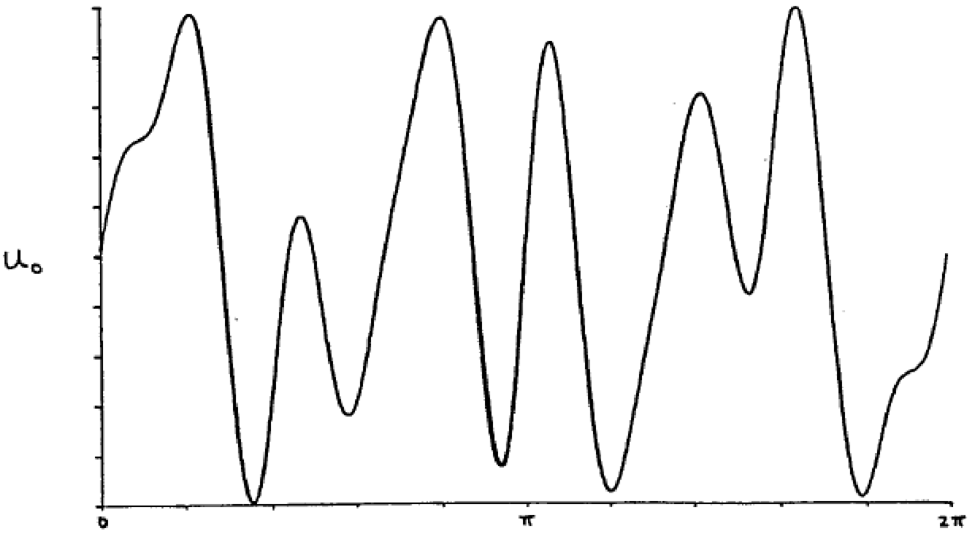


Fig. IV.5 Apparatus signal corresponding to the following values of the parameters of the Mueller polarimeter: $\alpha_2 = 45^\circ$, $\theta_1 = 0^\circ$, $\theta_2 = 90^\circ$, $\delta_1 = \delta_2 = 90^\circ$, $g_1 = g_2 = 1$.

IV.4. Computerized Fourier analysis of the recorded signal.

In order to obtain the Fourier coefficients corresponding to a tabulated function $u_0(x)$ from a measurement record, we have used a similar algorithm to the proposed by A. Ralston and H. Wilf [46]. The required data by our subroutine for Fourier analysis are the following

1- Entire value for N, so that there is an array of 2N + 1 distributed homogeneously inside the angular range as

$$\frac{2k\pi}{2N+1}, \quad k = 0,1,2,...,2N \qquad \text{(IV.40)}$$

2- Values of the function u(x) for $0 \leq x \leq 2\pi$, arranged in intervals of $2\pi/(2N+1)$.

3- Order M of the highest frequency harmonic in the Fourier series so that $0 \leq M \leq N$.

In our case, M = 14 and the minimum number of required data points is 2M +1 = 29.

Chapter V

Dynamic method for the analysis of polarized light

The study presented in the precedent chapter can be particularized to obtain the Stokes vector associated with a light beam. We will see that this can be obtained by means of a calibration of the device for the wavelength of the beam and a record of the intensity signal of the light beam after passing through the analyzing branch of the instrument (composed of L_2 and P_2).

V.I. Analysis device

In section IV.I we have seen that the device for the determination of Mueller matrices, schematized in Fig. IV.I, can be considered as divided into two parts. One of them, which contains L_2 and P_2, is used for the analysis of the state of polarization of the light that passes through it. The scheme of the Fig. V.I. shows the device used for the obtainment of the Stokes parameters corresponding to the light beam under study.

The Mueller matrix corresponding to the system formed by L_2 and P_2 is the matrix **B** given in (IV.7). Given an incoming light beam whose Stokes vector **S** is to be measured, the Stokes vector **U** corresponding to the light beam that emerges from the device is

$$\mathbf{U} = \mathbf{BS} \tag{V.1}$$

and the intensity of the emerging light is given by

$$u_0 = b_{00}s_0 + b_{01}s_1 + b_{02}s_2 + b_{03}s_3 \tag{V.2}$$

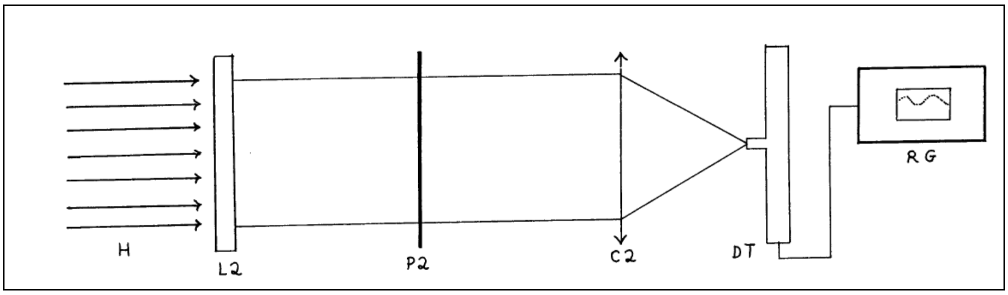


Fig.V.I: Scheme of the dynamic device for the analysis of polarized light.

H: collimated beam of monochromatic light
L2: rotatory retarder
P2: linear polarizer
C2: collector lens
DT: detector
RG: recorder

From (IV.7) and (V.2), and by using some trigonometric relations, we obtain [20]

$$u_0 = A_0 + B_1 \sin\omega + A_1 \cos\omega + B_2 \sin 2\omega + A_2 \cos 2\omega \qquad \text{(V.3)}$$

where

$$\omega = 2\omega_2$$

$$A_0 = (1+k_2)s_0 + \left[\frac{1}{2}(1+k_2) + r_2\cos\delta_2\right](s_1\cos 2\theta_2 + s_2\sin 2\theta_2)$$

$$B_1 = (1-k_2)\left[s_0\sin 2(\theta_2+\alpha_2) + s_1\sin 2\alpha_2 + s_2\cos 2\alpha_2\right] -$$
$$-2r_2\sin\delta_2 s_3\cos 2(\theta_2+\alpha_2)$$

$$A_1 = (1-k_2)\left[s_0\cos 2(\theta_2+\alpha_2) + s_1\cos 2\alpha_2 - s_2\sin 2\alpha_2\right] + \qquad \text{(V.4)}$$
$$+2r_2\sin\delta_2 s_3\cos 2(\theta_2+\alpha_2)$$

$$B_2 = \left[\frac{1}{2}(1+k_2) - r_2\cos\delta_2\right] + \left[s_1\sin 2(\theta_2+2\alpha_2) + s_2\cos 2(\theta_2+2\alpha_2)\right]$$

$$A_2 = \left[\frac{1}{2}(1+k_2) - r_2\cos\delta_2\right] + \left[s_1\cos 2(\theta_2+2\alpha_2) - s_2\sin 2(\theta_2+2\alpha_2)\right]$$

The two angular parameters θ_2, α_2 have not been specified in the previous expressions. In order to concrete, we will consider a Cartesian system of reference axes XYZ, so that the light propagates on the Z axis direction, and the polarization axis of the linear polarizer P_2 coincides with the X axis. With this choice, $\theta_2 = 0$, and α_2 being the angle formed by the fast axis of L_2 and the axis X at the initial instant, the expressions (V.4) are transformed into

$$A_0 = (1+k_2)s_0 + \left[\frac{1}{2}(1+k_2) + r_2\cos\delta_2\right]s_1$$

$$B_1 = (1-k_2)\left[s_0\sin 2\alpha_2 + s_1\sin 2\alpha_2 + s_2\cos 2\alpha_2\right] - 2r_2\sin\delta_2 s_3\cos 2\alpha_2$$

$$A_1 = (1-k_2)\left[s_0\cos 2\alpha_2 + s_1\cos 2\alpha_2 - s_2\sin 2\alpha_2\right] + 2r_2\sin\delta_2 s_3\sin 2\alpha_2 \qquad \text{(V.5)}$$

$$B_2 = \left[\frac{1}{2}(1+k_2) - r_2\cos\delta_2\right] + \left[s_1\sin 4\alpha_2 + s_2\cos 4\alpha_2\right]$$

$$A_2 = \left[\frac{1}{2}(1+k_2) - r_2\cos\delta_2\right] + \left[s_1\cos 4\alpha_2 + s_2\sin 4\alpha_2\right]$$

By means of the Fourier analysis of the intensity signal U_0, the Fourier coefficients of the series (V.3) can be obtained. If we know the parameters δ_2, k_2, α_2, which are characteristic of the analysis device, it is easy to prove from (V.5) that the elements of the Stokes vector **S** corresponding to the light beam under study are

$$
\begin{aligned}
s_1 &= \frac{B_2 \sin 4\alpha_2 + A_2 \cos 4\alpha_2}{\frac{1}{2}\left(1+k_2\right) - r_2 \cos\delta_2} \\
s_2 &= \frac{B_2 \cos 4\alpha_2 - A_2 \sin 4\alpha_2}{\frac{1}{2}\left(1+k_2\right) - r_2 \cos\delta_2} \\
s_3 &= \frac{A_1 \sin 2\alpha_2 - B_1 \cos 2\alpha_2 - \left(1-k_2\right) s_2}{2 r_2 \sin\delta_2} \\
s_0 &= \frac{A_0 - \left[\frac{1}{2}\left(1+k_2\right) + r_2 \cos\delta_2\right] s_1}{1+k_2}
\end{aligned}
\tag{V.6}
$$

V.2. Calibration

A procedure to obtain the parameters δ_2, k_2, α_2, is the obtainment of a record when a beam of X - linear polarized light falls on the device. In this case

$$
\mathbf{S} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}
$$

and the Fourier coefficients corresponding to the Fourier analysis of the intensity signal U_0 are

$$
\begin{aligned}
&A_0 = \frac{A_0'}{f} = \frac{3}{2}\left(1+k_2\right) + r_2 \cos\delta_2 \\
&B_1 = \frac{B_1'}{f} = 2\left(1-k_2\right)\sin 2\alpha_2 , \quad A_1 = \frac{A_1'}{f} = 2\left(1-k_2\right)\cos 2\alpha_2 \\
&B_2 = \frac{B_2'}{f} = \left[\frac{1}{2}\left(1+k_2\right) - r_2 \cos\delta_2\right]\sin 4\alpha_2 , \quad A_2 = \frac{A_2'}{f} = \left[\frac{1}{2}\left(1+k_2\right) - r_2 \cos\delta_2\right]\cos 4\alpha_2
\end{aligned}
\tag{V.7}
$$

where the coefficients A_i', B_i' are affected by a global scale factor *f* introduced by the instrument response.

The parameters Z_1, Z_2, h, defined as

$$Z_1 \equiv B_2' \sin 4\alpha_2 + A_2' \cos 4\alpha_2 + A_0' = 2f\left(1+k_2\right)$$
$$Z_2 \equiv B_1' \sin 2\alpha_2 + A_1' \cos 2\alpha_2 = 2f\left(1-k_2\right) \qquad \text{(V.8)}$$
$$h = Z_2 / Z_1$$

let us to obtain δ_2, k_2, α_2, f as follows

$$\tan 4\alpha_2 = B_2'/A_2'$$
$$k_2 = \left(1-h\right)/\left(1+h\right)$$
$$f = \frac{Z_1}{2\left(1+k_2\right)} \qquad \text{(V.9)}$$
$$\cos\delta_2 = \frac{\dfrac{A_0'}{f} - \dfrac{3}{2}\left(1+k_2\right)}{r_2}$$

To avoid indeterminations in the expressions (V.6) we have to impose the condition $\delta_2 \neq 0, \pi$

As we have just seen, the calibration of the analysis device is obtained by the realization of a record with a beam of X - linear polarized light falling on the device. By means of the computerized Fourier analysis of the recorded signal, the Fourier coefficients are obtained, and from them we calculate the parameters of the device according to (V.9).

The Fourier analysis of the signal is made in a similar way to that shown in section IV.5. The only difference is the existence of two harmonic terms, so that M = 2. Thus, the minimum number of data-points required by the computer program of Fourier analysis is 5.

V.3. Sensitivity of the apparatus signal with respect to the calibration parameters

In order to appreciate the influence of the several calibration parameters in the apparatus signal we have studied it systematically by varying each of the calibration parameters, and fixing the remainder.

Figures V.2, V.3 and V.4 show the recorded signal obtained by the variations of the parameters α_2, δ_2 and k_2, respectively (dotted line), in relation with the apparatus signal corresponding to the values $\alpha_2 = 0º$, $\delta_2 = 90º$ and $k_2 = 1$ (continuous line). From them we deduce that a variation of the value α_2 implies a global translation of the signal (Fig. V.2 shows a variation of +3º in α_2); the decrease of the value δ_2 is followed by an increase of the minimums (Fig. V.3), but the increase of the value δ_2 produces a grater elongation of the signal. Finally, the variation of the parameter k_2 generates a significant difference among the values for every two consecutive maximums (Fig. V.4).

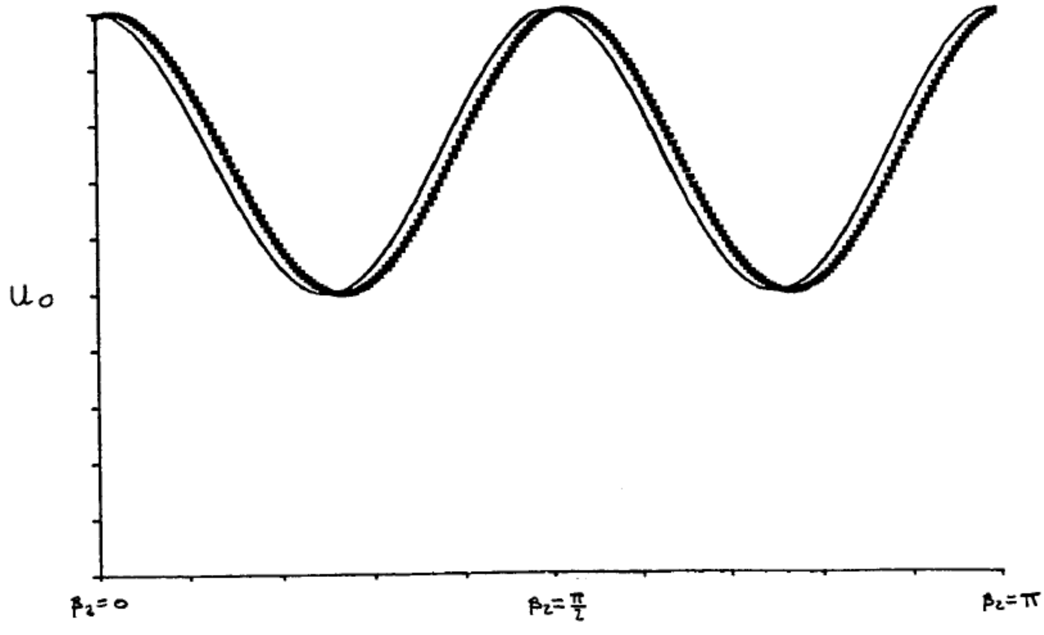


Fig. V.2: Apparatus signals corresponding to the following values of the parameters of the device for the analysis of polarized light

a) Continuous line: $\alpha_2 = 0º$, $\delta_2 = 90º$, $k_2 = 1$

b) Dotted line: $\alpha_2 = 3º$, $\delta_2 = 90º$, $k_2 = 1$

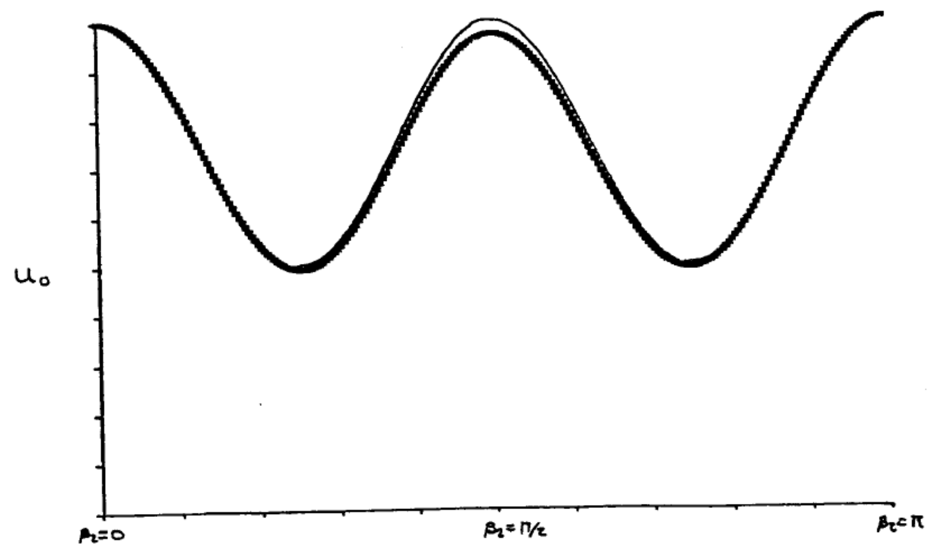


Fig. V.3: Apparatus signals corresponding to the following values of the parameters of the device for the analysis of polarized light

a) Continuous line: $\alpha_2 = 0°$, $\delta_2 = 90°$, $k_2 = 1$

b) Dotted line: $\alpha_2 = 0°$, $\delta_2 = 87°$, $k_2 = 1$

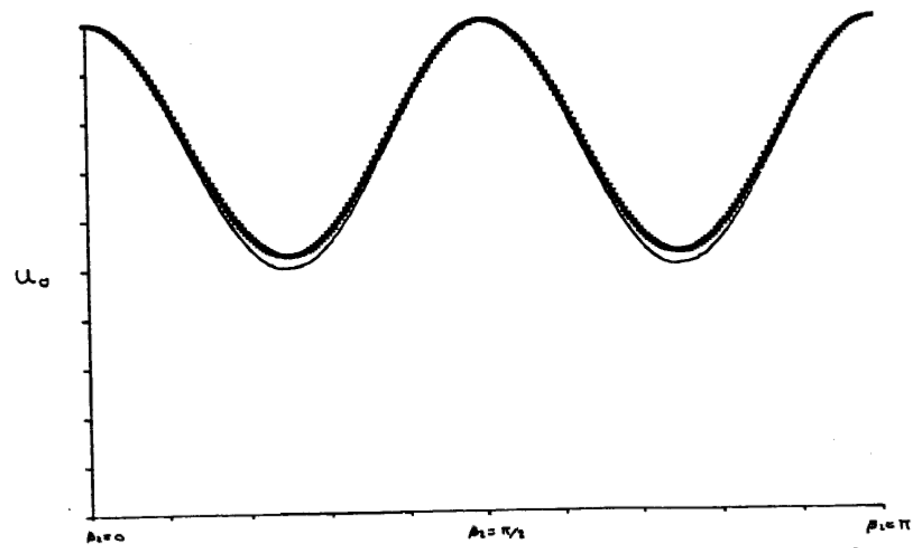


Fig. V.4: Apparatus signals corresponding to the following values of the parameters of the device for the analysis of polarized light

a) Continuous line: $\alpha_2 = 0°$, $\delta_2 = 90°$, $k_2 = 1$

b) Dotted line: $\alpha_2 = 0°$, $\delta_2 = 90°$, $k_2 = 0.97$

Chapter VI

Experimental device

The dynamic methods for the determination of Mueller matrices and Stokes parameters described in chapters IV and V require an adequate experimental device for a practical use. We have developed and performed an experimental assembly that let us the determination of the Mueller matrices associated with optical media operating by transmission. By the suppression of one of the rotatory retarders, the same assembly is also valid for the determination of the Stokes parameters of the studied light beam.

VI.1. General experimental assembly

Fig. VI.1 shows a general scheme of the experimental assembly used as Mueller polarimeter*.

Now, we make a numeration and description of the components of the device.

(a) He-Ne Laser
(b) Retarder L_1
(c) Mechanism producing the rotation of the retarders
(d) Studied optical medium $\mathcal{O}$, whose Mueller matrix **M** is to be measured
(e) Retarder L_2
(f) Total linear polarizer P_2
(g) Diaphragm F_1
(h) Diaphragm F_2
(i) Neutral filter N
(j) Diffuser DF
(k) Interferential filter FI
(l) Electronic device DE, which allows us to determine the origin and period of the recorded signal
(m) Detector DT
(n) Recorder
(o) Computer

** This experimental set-up is designed to be applied to optical media operating by transmission, but it can be easily adapted to the study of reflecting and scattering media.*

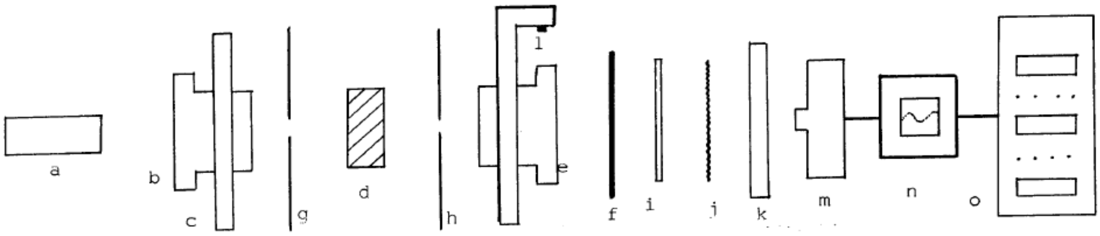


Fig. VI.1: Scheme of the experimental assembly used for the determination of Mueller matrices and the analysis of polarized light.

a) He-Ne Laser
b) Linear retarder
c) Mechanism producing the rotation of the retarders
d) Studied optical medium
e) Linear retarder
f) Total linear polarizer
g) Diaphragm
h) Diaphragm
i) Neutral filter
j) Diffuser
k) Interferential filter
l) Electronic device for the determination of the origin and period
m) Detector
n) Recorder
o) Electronic computer

As source of light we have used a Spectra-Physics He-Ne laser, model 120 A (λ = 632.8 nm).

The using of a laser light beam without expansion as test beam allows us the realization of a local exploration of the sample placed in the device.

We do not need to use any linear polarizer P_1 because the light from the laser is linearly polarized. The linear polarizer P_2 is a Polaroid HN-22 sheet, whose nominal values for the principal coefficients of the transmission in intensity for λ = 650 nm are

$$k = 0.48$$

$$k' = 2 \cdot 10^{-6}$$

and consequently

$$k_2 = \frac{k'}{k} \approx 4 \cdot 10^{-6}.$$

This justifies the theoretical assumption of P_2 as a total linear polarizer. Moreover, as the orientation of P_2 remains fixed on each measurement, there are not systematical errors originated by the different response of the detector for different states of polarization of the light falling on it. This effect has been proved in laboratory with several detectors.

We have used Polaroid commercial sheets as the retarders L_1, L_2. These sheets have a retardation nominal value of 140 ± 20 mμ, for a wavelength of λ = 560 nm.

The functionality of the diaphragms F_1, F_2 is to avoid the production of multiple reflections among the several components passed through by the light beam. These reflections would produce parasitic light beams falling on the detector.

The neutral filter N and the diffuser DF are used to decrease the intensity of light falling on the detector, in the case of being a photomultiplier.

The functionality of the interferential filter is to avoid the falling on the detector of ambient light with a wavelength different from the laser one.

A scheme of the mechanism that produces the rotation of the retarders L_1 and L_2 is shown in Fig. VI.2. This mechanism consists of a connection between two exterior parallel gears, which are moved by a non-synchronic motor (MA) placed against the front face of the aluminum holder used as support of the device. The motor, with an approximate power of 200 W, has a velocity of 3.500 r.p.m. This motor transmits the movement to the steel axis (EA) by means of a parallel gear (EP).This axis rotates around little bearings, and two motor pinions, (PM1) and (PM2), rotates with it. The resistant wheels of the gear are the cogged wheels (CD1) and (CD2), which rotate with two cylindrical screws placed on the holder-sheets (PS1) and (PS2). These screws rotate on two connections of two ball bearings. We have put these ball bearings in pairs to avoid lateral movement of the screws. These screws sticks out the exterior faces, and the carcasses (CS1) and (CS2) are against them, used as holder for the retardation sheets. By means of a set of three screws we can regulate the perpendicularity of each sheet with the optical axis of the system.

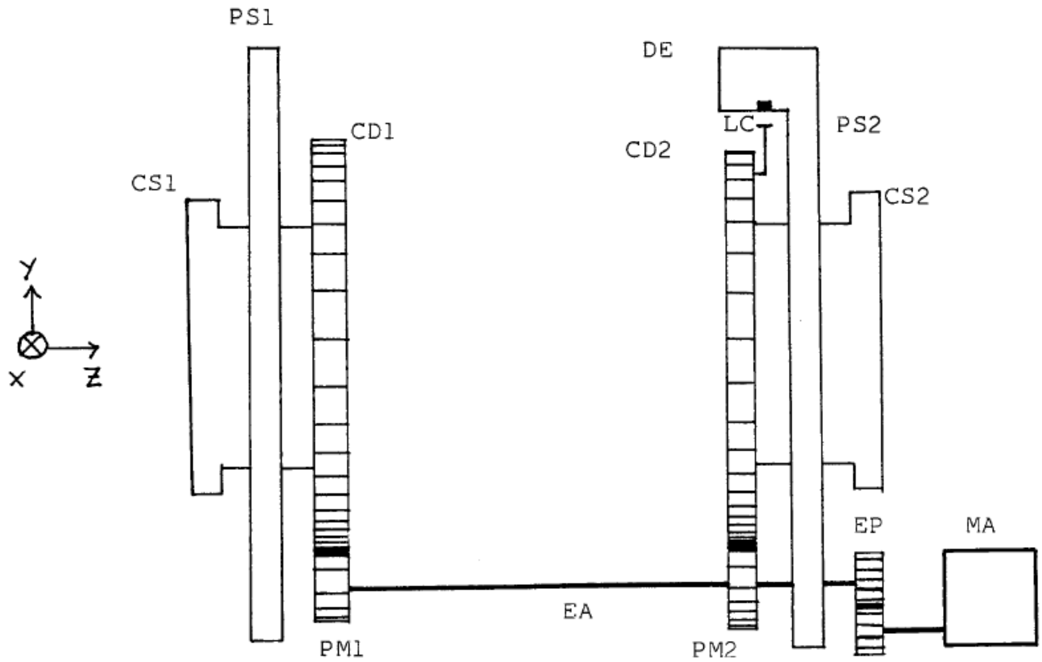


Fig. VI.2: Scheme of the mechanism that generates the rotation of the retarders.

MA: Non-synchronic motor
EA: Steel axis
EP: Parallel gear
PM1, PM2: Motor pinions
CD1, CD2: Cogged wheels
PS1, PS2: Holder sheets
CS1, CS2: Carcasses used as holder of the retarder
LC: Cooper sheet
DE: Shoot electronic device, periodically activated by LC

To keep the consistence of the machinery, the sheets (PS1) and (PS2) are united by other two lateral sheets (PL1) and (PL2). The set is fixed by means of four “legs” to the base of the optical bench.

The intermediate space between the cogged wheels let us the placing of the optical medium to be studied, which can be moved on the XY plane so that its properties in several points can be studied.

In order to fix the origin of the detected signals, the electronic device schematized in Fig. VI.3 has been designed, which contains a coil where Foucault currents are induced when a conductor is moved near it. The coil is connected to a flip-flop system that transforms the peaks into a squared-signal.

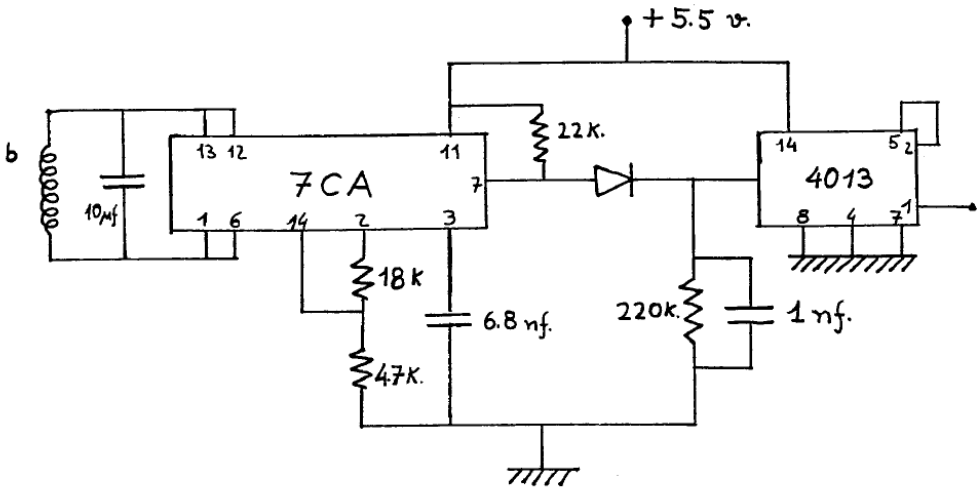


Fig. VI.3: Scheme of the electronic device for the determination of the origin and period of the recorded signals.

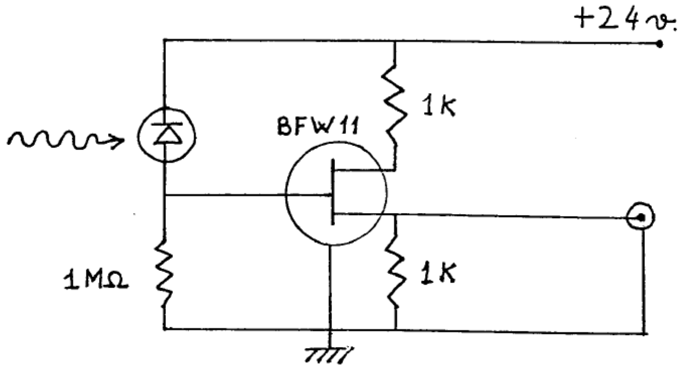


Fig. VI.4: Scheme of the circuit used to polarize the detector photodiode Harsaw 38.

Fixed to the cogged wheel CD2, there is a small cooper sheet (LC), which automatically activates the recorder when it passes by the place where the coil is. Thus, this device is periodically activated with the same period as the signal of the intensity of the light that falls on the detector DT. The period of time between two equivalent positions of the measurement device is T ~ 0.1 s.

We have used a photomultiplier Oriel 7060 as detector DT. It is necessary to decrease the laser light beam with the help of a neutral filter N and the diffuser DF so that DT works in linear response regime. Other detector also used is the photodiode Harsaw 38, polarized according to the circuit shown in Fig. VI.4. This kind of detector, although cheaper, introduce a certain level of continuous voltage over the detected signal. This level can be easily determined and must be subtracted from the recorded signal.

The recorder used is a multichannel analyzer HP 648OB, which records the signal from the detector and starts the records according to the rhythm of the shoot electronic device DE.

The work wavelength (λ = 632.8 nm) does not correspond to the optimum response zone of the photomultiplier used here, and thus we need to filter the random noise originated by the photons falling on it. This is made by means of a number of 2^8 signal scans, averaged with an algorithm of the multichannel analyzer. This number of scans has been considered adequate to obtain a signal/noise ratio high enough (s/N $\geq$ 100).

We have used a computer HP2000 for the processing of the signal data.

VI.2 Possible causes of errors in the measurements

In this section we analyze the principal possible causes of errors in the measurements obtained with our experimental device.

- Depolarization of the test light beam [2].

  In the theoretical treatment of our dynamic methods of measurement, we haven't taking into account the possible effect of depolarization of the test light beam produced by the several elements of the device, as the rotatory retarders and the linear polarizer P_2. Small specks of dust or imperfections on the surface of the elements can produce diffraction of the light beam, and thus a slight effect of depolarization.

- Deviations from the direction of the light beam

  The lack of perpendicularity of the surfaces of the elements with respect to the propagation direction of the light beam provokes a behavior of these elements different from the theoretically predicted. Otherwise, the angle of incidence over them and the polarizer P_2 can change constantly, although periodically, because of the rotation of the retarders during the measurement.

We have yet said that the mechanism for the rotation in the retarders has a device for the adjustment of the perpendicularity of the retardation sheets respect to the propagation direction of light. However, it is no easy to make the adjustment with great perfection because of the fact that the sheets are not perfectly flat, with small inhomogeneities and a certain global curvature. Otherwise, a perfect adjustment is not desirable because it would produce an overlapping of the direct light beam with the several reflected beams, which would fall over the detector and would produce a deterioration of the quality of the signal. This overlapping is worse than a little defect on the perpendicularity of the retardation sheets [47, 48], which allows us the elimination of parasitic beams by means of diaphragms.

The lack of perpendicularity between the retardation sheet L_1 and the direction of light propagation has the consequence of a periodic variation of the orientation of the own axis respect to the impact plane, with the same period of the signal $T = 2\pi/\omega_1$. This provokes that the effective values δ_1 and k_1 also vary periodically. The effect produced by the rotation of L_2 is more complicated, because the light beam that falls on it varies its angle respect to the axis Z with a period $2\pi/\omega_1$ and the orientation of the own axis of L_2 respect to a fixed axis varies with a period $T_2 = 2\pi/\omega_2 = 4\pi/5\omega_1 = 2T/5$. And thus, as $2\ T = 5\ T_2$, the situation is repeated with a double period for each signal.

One way to estimate the influence of this phenomenon, and to adjust the tilt of the retarders so that we obtain a minimum influence, is to check visually, on the screen of the multichannel, the differences in the shape of two consecutive periods.

- Multiple internal reflection in L_1 and L_2

  This effect is included in the theory, by considering the retarders L_1 and L_2 as non ideal. The effective values of δ_1, δ_2, k_1 and k_2 are obtained by means of the calibration.

- Imperfections on the mechanical device.

  The mechanism for the rotation of the sheets is subject to vibrations during the measurements and it can produce some disruption reflected in the results.

- Calibration errors.

  We must carefully calibrate the device because the errors in the characteristic parameters are transmitted systematically to all the measurements.

- Detection errors.

The dependence of the sensitivity of the detector with the polarization of the light does not become apparent, because the position of the polarizer P_2 remains fixed during each measurement. However, we must be sure that the detector is working on its linear response zone.

- Errors produced during the processing of the data.

  We have seen that the errors produced in the process of the computerized calculations are of the order of 0.05 % [22] and, as we will see in the next chapter, the introduction of a high number of data-points of the recorded signal in the Fourier analysis subroutine does not give us advantages in the quality of the measurements.

VI.3. Computerized data processing

Each record obtained with the multichannel analyzer is composed of 1000 data-points, where approximately 713 correspond to a complete period of the recorded signal. By connecting the computer with a digital voltmeter and the multichannel analyzer we can insert the data by means of an interface HP-IB (mod. 82937 A), or we can also connect the computer with the detector.

Once in the computer, the data are stored in files. There are four types of programs for the treatment of these data.

- MAPAR program for the treatment of apparatus signals obtained with the device for the determination of Mueller matrices.

- MEREL program for the treatment of the signal corresponding to the studied optical media.

- MEPOL program for the treatment of apparatus signals obtained with the device for the analysis of polarized light.

- STOKES program for the treatment of signals corresponding to studied beams of light.

VI.3.1. MAPAR program

The aim of this program is the obtainment of the calibration parameters of the device for the determination of the Mueller matrices.

Required data:

- Name of the data file corresponding to the apparatus-signal to be processed.
- Number NP of data-point to be inserted into the subroutine for the Fourier analysis.
- Continuous level introduced in the signal by the detector, if it exists.

The MAPAR program has three parts:

-MAPAR.1.

- By means of a linear interpolation, NP data points are selected among the total inserted in the file. This interpolation is justified by the relative proximity between two data-points of the record, and it needs 2NP data-points, from which are calculated the NP interpolated data.

-MAPAR.2.

- The interpolated data-points are inserted into the subroutine AJTE of Fourier analysis, and the Fourier coefficients A_i, B_j are obtained.

-MAPAR.3.

- The characteristic parameters of the device δ_1, δ_2, k_1, k_2, θ_1, θ_2, α_2 are calculated from the Fourier coefficients A_i, B_j,. It is easy to observe that none of these parameters depend on the scale factor that affects the signal.

VI.3.2. MEREL program

This program is used to obtain the Mueller matrix **M** associated with the measured optical system.

Required data:

- Name of the data file corresponding to the signal to be processed.
- Number NP of data-point to be inserted into the subroutine for the Fourier analysis.
- Continuous level introduced in the signal by the detector, if it exists.
- Values of the parameters δ_1, δ_2, k_1, k_2, θ_1, θ_2, α_2.

The MEREL program has three parts or subprograms. MEREL.1 and MEREL.2 are analogous to MAPAR.1 and MAPAR.2.

-MEREL.3.

- From the coefficients A_i, B_j, calculated in MEREL.2, the elements m_{ij} of the Mueller matrix **M** are calculated. These elements are obtained by means of the expressions (IV.20), and are affected by a unique factor, related with the scale factor of the signal. The program gives as a result the Mueller matrix $\mathbf{M_N}$, normalized as $\mathbf{M_N} = \mathbf{M}/m_{00}$

The programs MAPAR and MEREL are prepared to represent graphically the signal points.

VI.3.3. MEPOL and STOKES programs

The aim of the MEPOL program is analog to the MAPAR one, and the type of required data is the same. The MEPOL program let us make a calibration of the device for the analysis of polarized light, by calculating the parameters δ_2, k_2, α_2 with the equations (V.9).

The Stokes program is analog to MEREL and let us to obtain the Stokes vector **S** associated with the studied light beam. This vector is obtained by normalizing **S** so that $s_0 = 1$.

Chapter VII

Calibration and some results

In this chapter we present the experimental results corresponding to the calibration of the Mueller and Stokes polarimeters as well some measurements of Stokes vectors and Mueller matrices. These results are analyzed and discussed with the help of several relations and theorems included in chapters II and III. The calibration measurements are compared with the ideal theoretical results, so that the precision of the instrument is studied and compared with the obtained by means of other kind of methods and dynamic and static measurement devices.

The experimental assemblies used for the analysis of polarized light and the determination of Mueller matrices are described in the previous chapter.

All the results presented in this chapter have been obtained from intensity signals detected by a photomultiplier and recorded by making an average of 2^8 scans with the multichannel analyzer, as indicated in chapter VI.

VII.1. Determination of Stokes parameters

VII.1.1. Calibration

According to the expressions (V.9) we have made a calibration of the device used for the determination of Stokes parameters. In the data processing made by the computer we have seen that the obtained values for the Fourier coefficients do not change significantly for different numbers of data-points. The measured Fourier coefficients are

$$A_0' = 5.804, \quad A_1' = -0.088, \quad B_1' = -0.066, \quad A_2' = 0.834, \quad B_2' = 1.902 \qquad \text{(VII.1)}$$

which lead to the following parameters of the device

$$\alpha_2 = 106.6^{\circ} \qquad \delta_2 = 90.0^{\circ} \qquad k_2 = 0.974 \qquad \text{(VII.2)}$$

In order to make an estimation of the error produced during the calculation of the Stokes parameters we have considered that the coefficients (VII.1) correspond to the studied light beam, instead of assuming that the incoming light is linearly polarized along X direction. By considering (VII.2) as data we obtain the Stokes vector **S** corresponding to the beam by means of (V.6). The measured vector components are

$$s_0 = 1.000\,,\;\; s_1 = 1.000\,,\;\; s_2 = 0.000\,,\;\; s_3 = -0.002 \qquad \text{(VII.3)}$$

The degree of polarization of the light beam is

$$G = 1.000 \qquad \text{(VII.4)}$$

The results (VII.3-4) must be compared with the ideal values $s_0 = s_1 = 1$, $s_2 = s_3 = 0$, $G = 1$.

Fig. VII.1 shows the intensity signal obtained experimentally (dotted line) and the ideal theoretical signal corresponding to the values (VII.2) (continuous line).

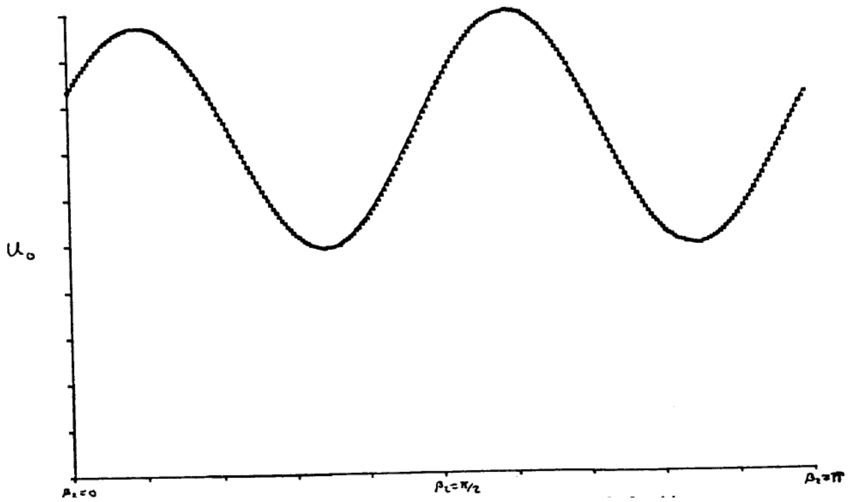


Fig. VII.1: Calibration apparatus-signal of the Stokes polarimeter with values $\alpha_2 = 106.6^\circ$, $\delta_2 = 90.0^\circ$, $k_2 = 0.974$. Experimental signal (dotted line) versus ideal theoretical signal for the same values (continuous line).

VII.1.2. Elliptical polarization

To illustrate the behavior of the Stokes polarimeter we have analyzed a light beam with a certain state of elliptical polarization.

In this particular case, the obtained Fourier coefficients are

$$A_0 = 1.844,\;\; A_1 = -0.889,\;\; B_1 = 1.313,\;\; A_2 = -0.141,\;\; B_2 = 0.282 \qquad \text{(VII.5)}$$

which correspond to the following Stokes parameters

$$s_0 = 1.000, \quad s_1 = 0.231, \quad s_2 = 0.275, \quad s_3 = 0.930 \tag{VII.6}$$

The degree of polarization of the light beam is

$$G = 0.997 \tag{VII.7}$$

In Fig. VII.2 we see the shape of the recorded intensity signal.

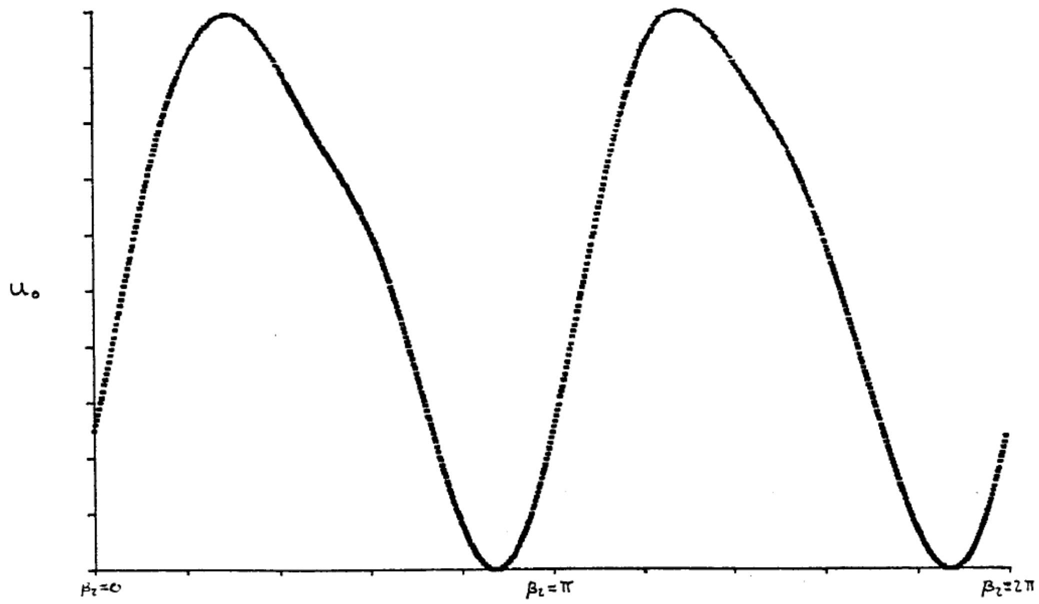


Fig. VII.2: Experimental signal corresponding to an elliptically polarized light beam with Stokes parameters $s_0 = 1.000$, $s_1 = 0.231$, $s_2 = 0.275$, $s_3 = 0.930$.

VII.2. Determination of Mueller matrices

VII.2.1. Calibration

We have made a calibration of the Mueller polarimeter as indicated in section IV.2.

The Fourier coefficients are obtained from the analysis of the calibration signal

$$
\begin{aligned}
&A_0 = 1.735 \\
&A_1 = -0.026 \qquad B_1 = 0.006 \\
&A_2 = -0.008 \qquad B_2 = -0.015 \\
&A_3 = -1.065 \qquad B_3 = 0.085 \\
&A_4 = -0.068 \qquad B_4 = -0.462 \\
&A_5 = 0.007 \qquad B_5 = -0.015 \\
&A_6 = -0.566 \qquad B_6 = -0.216 \\
&A_7 = -0.293 \qquad B_7 = 1.038 \\
&A_8 = -0.007 \qquad B_8 = 0.011 \\
&A_9 = -0.006 \qquad B_1 = 0.011 \\
&A_{10} = 0.435 \qquad B_{10} = -0.474 \\
&A_{11} = -0.001 \qquad B_{11} = 0.027 \\
&A_{12} = 0.006 \qquad B_{12} = 0.000 \\
&A_{13} = -0.003 \qquad B_{13} = -0.005 \\
&A_{14} = 0.005 \qquad B_{14} = -0.002
\end{aligned}
\tag{VII.8}
$$

From these coefficients, the following parameters of the device are obtained

$$
\begin{aligned}
&\theta_1 = 27.8^{\circ}, \ \ \theta_2 = -77.0^{\circ}, \\
&\alpha_2 = 77.4^{\circ}, \\
&\delta_1 = 88.4^{\circ}, \ \ \delta_2 = 92.5^{\circ}, \\
&k_1 = 0.981, \ \ k_2 = 0.982 .
\end{aligned}
\tag{VII.9}
$$

To estimate the error in the values of the elements of the measured Mueller matrices, we have considered the Fourier coefficients of the apparatus-signal as the corresponding to a Mueller measurement, in such a manner that we have obtained the Mueller matrix from (IV.23), using (VII.9) as the values of the parameters of the device. This measured Mueller matrix is

$$
\mathbf{M} = \begin{pmatrix} 1.000 & 0.004 & 0.000 & 0.002 \\ -0.004 & 1.021 & 0.003 & 0.007 \\ 0.002 & 0.003 & 1.004 & -0.007 \\ -0.009 & 0.000 & 0.008 & 0.998 \end{pmatrix}
\tag{VII.10}
$$

For an ideal and perfect measurement device, the Mueller matrix obtained by means of this self-calibration procedure should be the identity matrix because the signal of light intensity has been obtained without any optical medium placed as a sample.

The values of the norm and the values of the polarization and depolarization indices corresponding to this measured matrix are

$$\Gamma_{\mathrm{M}} = 2.012, \qquad G_D = 0.992, \qquad G'_P = 0.009, \qquad G''_P = 0.012 \tag{VII.11}$$

A complete period of the intensity signal takes up 713 data-points in the record. The coefficients (VII.8) have been obtained by selecting 513 data-points by means of linear interpolation. This interpolation is justified by the proximity among the original data-points of the record.

As said in chapter VI, the minimum number of data-points to insert into the Fourier analysis subroutine is 29. By selecting only 29 data-points we obtain Fourier coefficients very similar to the (VII.8). The obtained matrix in this case is

$$\mathbf{M} = \begin{pmatrix} 1.000 & 0.000 & -0.005 & 0.000 \\ 0.000 & 1.018 & -0.010 & 0.002 \\ 0.006 & -0.009 & 1.000 & -0.010 \\ -0.009 & -0.003 & 0.007 & 0.998 \end{pmatrix} \tag{VII.12}$$

The comparison between (VII.10) and (VII.12) tells us that the number of data-points used for the calculation of Fourier coefficients does not affect significantly to the results, and we can consider these results as acceptable even for a minimum number of 29 data-points.

From the signals corresponding to four series of records made in similar conditions and by considering each apparatus-signal as the problem signal, we have calculated the respective Fourier coefficients, and from them we have obtained the respective calibration parameters as well as the corresponding Mueller matrices. By making a statistical study of the results we have obtained the following values

$$\begin{gathered} \theta_1 = 27.6^{\circ} \pm 0.8^{\circ}, \quad \theta_2 = -77.1^{\circ} \pm 1.0^{\circ}, \\ \alpha_2 = 77.1^{\circ} \pm 0.3^{\circ} \\ \delta_1 = 88.7^{\circ} \pm 0.6^{\circ}, \quad \delta_2 = 92.3^{\circ} \pm 0.6^{\circ} \\ k_1 = 0.978 \pm 0.006, \quad k_2 = 0.984 \pm 0.005 \end{gathered} \tag{VII.13}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 0.002 \pm 0.002 & -0.000 \pm 0.002 & -0.001 \pm 0.003 \\ -0.001 \pm 0.002 & 1.013 \pm 0.001 & -0.002 \pm 0.005 & 0.004 \pm 0.007 \\ 0.003 \pm 0.002 & -0.003 \pm 0.004 & 1.009 \pm 0.011 & -0.003 \pm 0.006 \\ -0.007 \pm 0.004 & 0.005 \pm 0.009 & 0.008 \pm 0.007 & 0.998 \pm 0.002 \end{pmatrix} \quad \text{(VII.14)}$$

These results reveal a good reproducibility on the measurements, with an average accidental error of approximately 0.5 % in the determination of the elements of the Mueller matrices.

By comparing (VII.14) with the identity matrix, which would be the result obtained with an ideal and perfect instrument, and taking into account the accidental and systematical errors, we can estimate a mean error less than 1 % in the results.

In Figures VII.3-VII.5 several apparatus-signals obtained experimentally (dotted line), are represented combined with the corresponding ideal apparatus-signals (continuous line). These graphics give visual information about the quality of the experimental set-up.

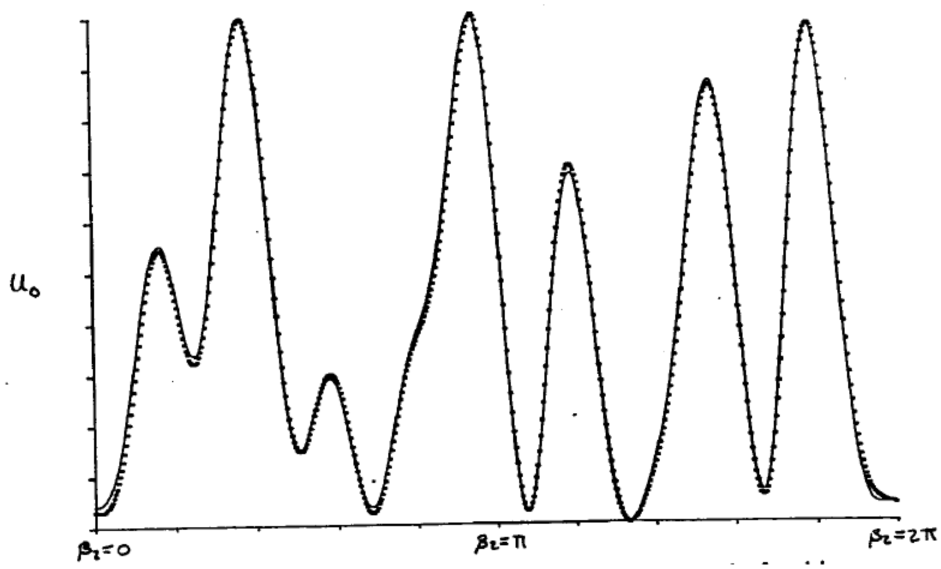


Fig. VII.3: Calibration apparatus-signal of the Mueller polarimeter, with values θ_1 = 27.8º, θ_2 = -77.0º, δ_1 = 88.4º, δ_2 = 92.5º, α_2 = 77.4º, k_1 = 0.981, k_2 = 0.982. Experimental signal (dotted line) versus ideal theoretical signal for the same values (continuous line).

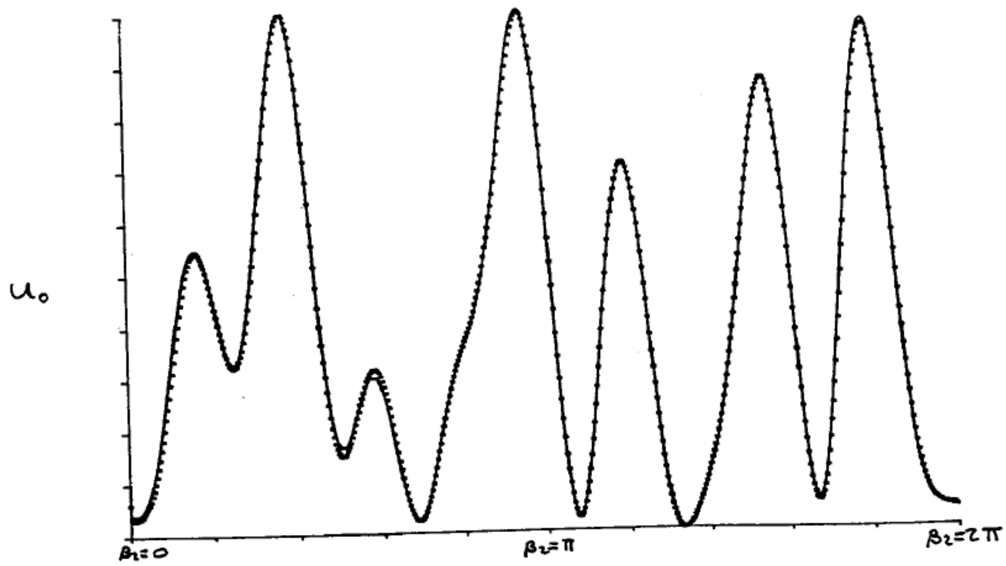


Fig. VII.4: Calibration apparatus-signal of the Mueller polarimeter, with values θ_1 = 28.5º, θ_2 = -78.5º, δ_1 = 88.4º, δ_2 = 91.5º, α_2 = 76.7º, k_1 = 0.971, k_2 = 0.980. Experimental signal (dotted line) versus ideal theoretical signal for the same values (continuous line).

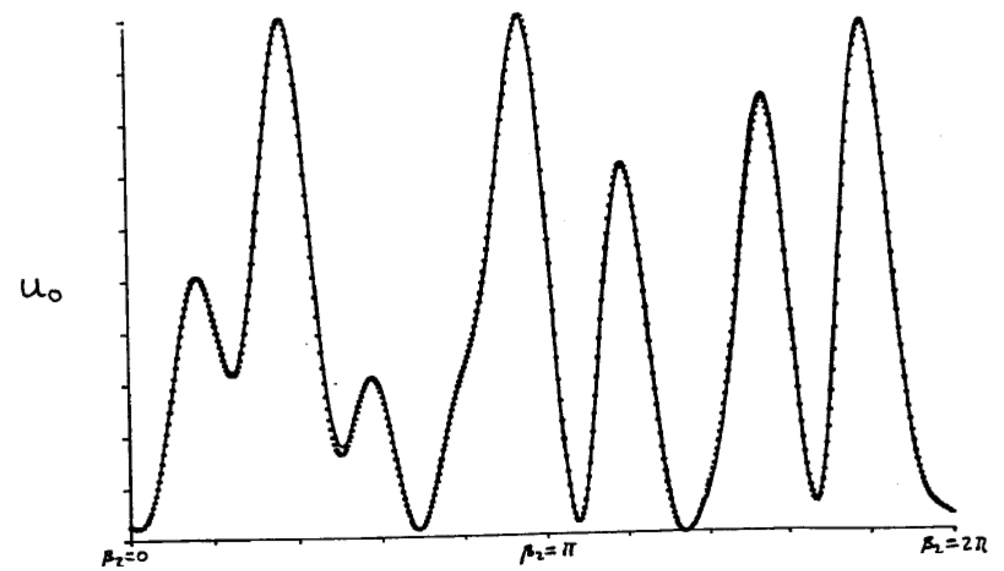


Fig. VII.5: Calibration apparatus-signal of the Mueller polarimeter, with values θ_1 = 26.6º, θ_2 = -75.9º, δ_1 = 89.6º, δ_2 = 92.9º, α_2 = 77.2º, k_1 = 0.966, k_2 = 0.991. Experimental signal (dotted line) versus ideal theoretical signal for the same values (continuous line).

VII.2.2. Commercial retarder

We present here the results corresponding to the measurement of the Mueller matrix associated with a Polaroid commercial linear retardation sheet, with a nominal retardation value of 140 ± 20 mμ for a wavelength of $\lambda = 589$ nm.

The Mueller matrix $\mathbf{M}_{\mathbf{L}_1}$ obtained for a certain orientation of the retarder axes respect to the reference axes is

$$\mathbf{M}_{\mathbf{L}_1} = \begin{pmatrix} 1.000 & 0.012 & 0.004 & 0.033 \\ 0.014 & 1.035 & -0.162 & 0.181 \\ 0.000 & -0.115 & 0.182 & 0.971 \\ 0.004 & -0.143 & -0.984 & 0.129 \end{pmatrix} \quad \text{(VII.16)}$$

from which we obtain

$$\Gamma_{\mathrm{M}} = 2.031, \qquad G_D = 0.983, \qquad G'_P = 0.014, \qquad G''_P = 0.036 \quad \text{(VII.17)}$$

In the case of neglecting the small effects of partial polarization and depolarization, that is, if we consider that the following conditions are fulfilled in (VII.16) m_{i0} , $m_{0i} \approx 0, \quad i = 1,2,3$; then, as we have seen in section II.4, the medium behaves as an ideal retarder.

From (III.30) we obtain

$$\Delta = 80.0^{\circ}, \qquad \omega = 1.2^{\circ}, \qquad \psi = -0.5^{\circ} \quad \text{(VII.18)}$$

The retarder preserves invariant the states of polarization with azimuth Ψ and ellipticity $\pm\omega$, introducing between them a retardation phase Δ.

Because of $|\omega| \approx 0$, the retarder behaves as a linear retarder.

With the help of (III.36.c-d) we can calculate the ratio k between the principal coefficients of transmission, and we obtain

$$k = 0.975 \quad \text{(VII.19)}$$

As expected, the value k is slightly different from the unity, because the retarder is not ideal.

The intensity signal corresponding to this measurement is represented in Fig. VII.6.

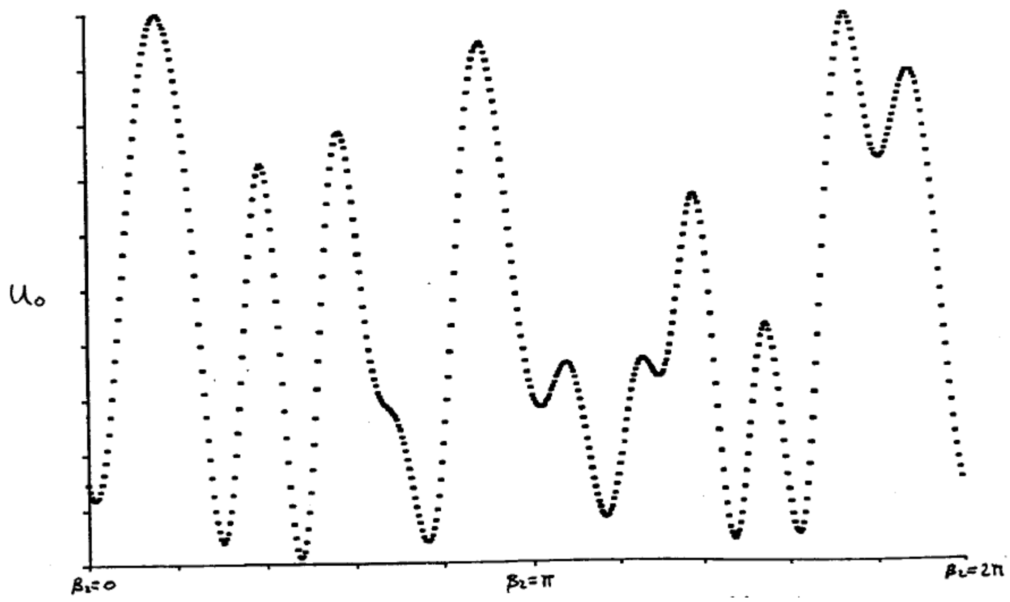


Fig. VII.6: Experimental signal corresponding to a commercial linear retarder with parameters Δ = 80.0º, Ψ= -0.5º, k = 0.975.

To study the operating of the device and the calculation apparatus, we have made another record with the same optical medium, but with a different orientation respect to the device. The measured Mueller matrix is

$$\mathbf{M}_{\mathbf{L}_2} = \begin{pmatrix} 1.000 & -0.010 & 0.000 & 0.019 \\ -0.016 & 0.335 & 0.402 & -0.852 \\ -0.002 & 0.360 & 0.799 & 0.525 \\ 0.006 & 0.875 & -0.460 & 0.114 \end{pmatrix} \quad \text{(VII.20)}$$

from where

$$\Delta = 82.8^{\circ}, \qquad \omega = 1.3^{\circ}, \qquad \psi = 30.2^{\circ} \quad \text{(VII.21)}$$

These results are in good accordance with (VII.18) because Δ and ω are not significantly different in both cases. The value of Ψ has changed because the orientation of the optical medium has changed.

VII.2.3. Commercial linear polarizer

In this case we have considered a Polaroid HN42 commercial linear polarizer as the sample under measurement.

The measured Mueller matrix $\mathbf{M_P}$ for a certain orientation of the polarization axis of the polarizer respect to the reference axes of the polarimeter is

$$\mathbf{M_P} = \begin{pmatrix} 1.000 & -0.856 & -0.668 & -0.018 \\ -0.864 & 0.685 & 0.520 & -0.007 \\ -0.675 & 0.534 & 0.413 & -0.003 \\ -0.007 & 0.045 & -0.015 & -0.005 \end{pmatrix} \quad \text{(VII.22)}$$

The measured values of the norm and indices corresponding to $\mathbf{M_P}$ are

$$\Gamma_{\mathrm{M}} = 2.033, \qquad G_D = 0.976, \qquad G'_P = 0.897, \qquad G''_P = 0.917 \quad \text{(VII.23)}$$

If the small effect of depolarization is neglected and considering that, according to (VII.22), $\mathbf{M_P^T} \approx \mathbf{M_P}$, $\left(m_{01}^2 + m_{02}^2 + m_{03}^2\right)^{1/2} \approx m_{00}$; then, in agreement with section III.4, the matrix $\mathbf{M_P}$ corresponds to a polarizer with parameters

$$\delta = 0.0^{\circ}, \quad \nu = 19.0^{\circ}, \qquad k = 0.041 \quad \text{(VII.24)}$$

These results indicates that this system is a linear polarizer ($\delta = 0^{\circ}$).

In Fig. VII.7, where the intensity signal corresponding to $\mathbf{M_P}$ is represented, we can see that (except for slight differences) the signal is doubly periodic. This is a characteristic property for systems whose last element is a total linear polarizer.

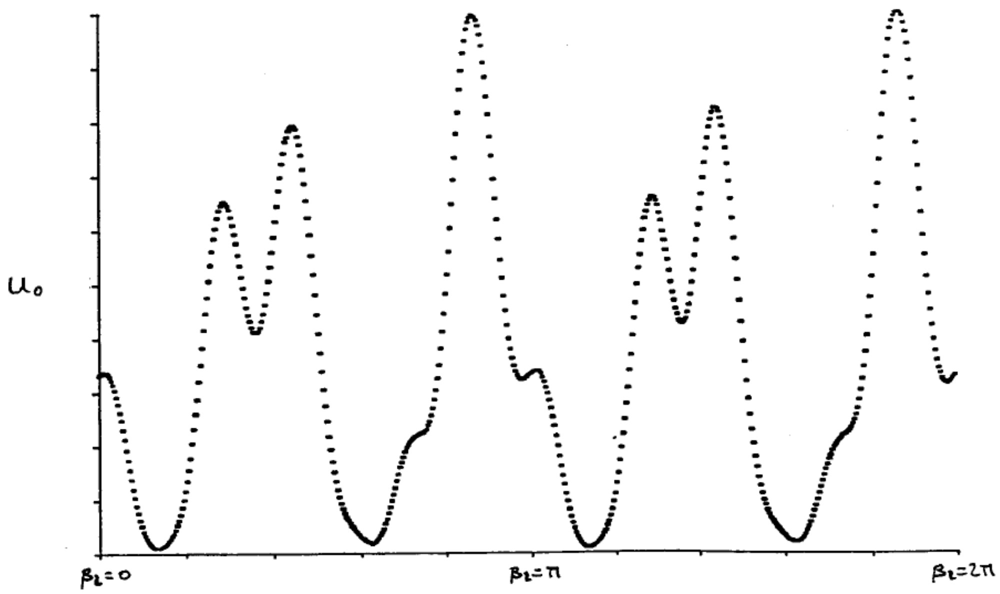


Fig. VII.7: Experimental signal corresponding to a linear polarizer with parameters $\nu = 19^\circ$, $k = 0.041$.

VII.2.4. System of two linear retarders.

We have obtained the Mueller matrix associated with a system composed of two Polaroid commercial quarter-wave linear retarders for a wavelength $\lambda = 598$ nm. For a certain position of the axes respect to the reference ones, the measured matrix is

$$\mathbf{M}_{2\mathrm{L}} = \begin{pmatrix} 1.000 & 0.002 & 0.001 & -0.031 \\ -0.008 & 0.659 & 0.057 & 0.850 \\ 0.004 & -0.320 & -0.905 & 0.215 \\ 0.044 & 0.759 & -0.375 & -0.507 \end{pmatrix} \qquad \text{(VII.25)}$$

so that

$$\Gamma_{\mathrm{M}} = 2.026, \qquad G_D = 0.982, \qquad G'_P = 0.045, \qquad G''_P = 0.031 \qquad \text{(VII.26)}$$

These results indicate that the behavior of the system is similar to the behavior of an elliptic retarder with parameters

$$\Delta = 151.2^\circ, \qquad \omega = 25.7^\circ, \qquad \psi = -4.3^\circ \qquad \text{(VII.27)}$$

The ratio k between the principal coefficients of intensity transmission is

$$k = 0.938 \tag{VII.28}$$

which tells us that the effect of partial polarization is bigger now than in the case with one retarder of the same kind.

The intensity signal corresponding to $\mathbf{M_{L2}}$ is represented in Fig. VII.8.

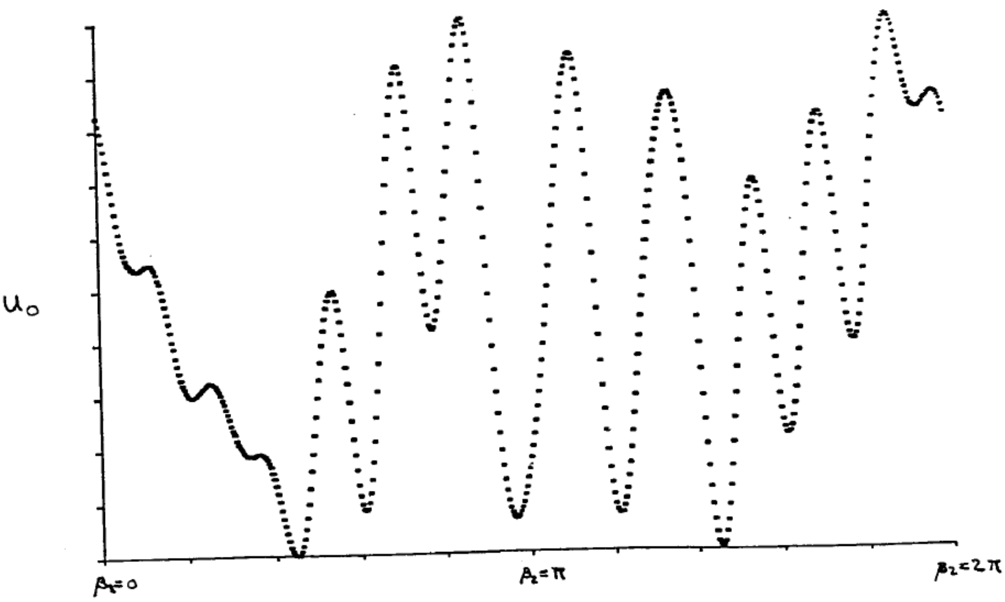


Fig.VII.8: Experimental signal corresponding to a system with two linear retarders equivalent to an elliptic retarder with parameters Δ = 151.2º, ω = 25.7º, Ψ = -4.3º, k = 0.938.

VII.2.5. System with three linear retarders

The Mueller matrix experimentally obtained for a system composed of three commercial linear retarders of the same kind as the analyzed in section VII.2.2, for a certain orientation of its own axes respect to the reference ones is

$$\mathbf{M}_{3L} = \begin{pmatrix} 1.000 & -0.035 & 0.019 & 0.014 \\ -0.002 & 0.619 & -0.205 & 0.777 \\ 0.048 & 0.817 & 0.163 & -0.607 \\ -0.032 & 0.027 & 0.990 & 0.313 \end{pmatrix} \tag{VII.29}$$

from where

$$\Gamma_M = 2.043, \qquad G_D = 0.970, \qquad G'_P = 0.042, \qquad G''_P = 0.062 \qquad \text{(VII.30)}$$

The matrix $\mathbf{M_{3L}}$ approximately corresponds to an ideal elliptic retarder with parameters

$$\Delta = 87.2^\circ, \qquad \omega = 15.4^\circ, \qquad \psi = 12.9^\circ \qquad \text{(VII.31)}$$

The effect of partial polarization, which produces a behavior different than the one of an ideal retarder, is given by

$$k = 0.919 \qquad \text{(VII.32)}$$

The shape of the intensity signal corresponding to this case is shown in Fig. VII.9.

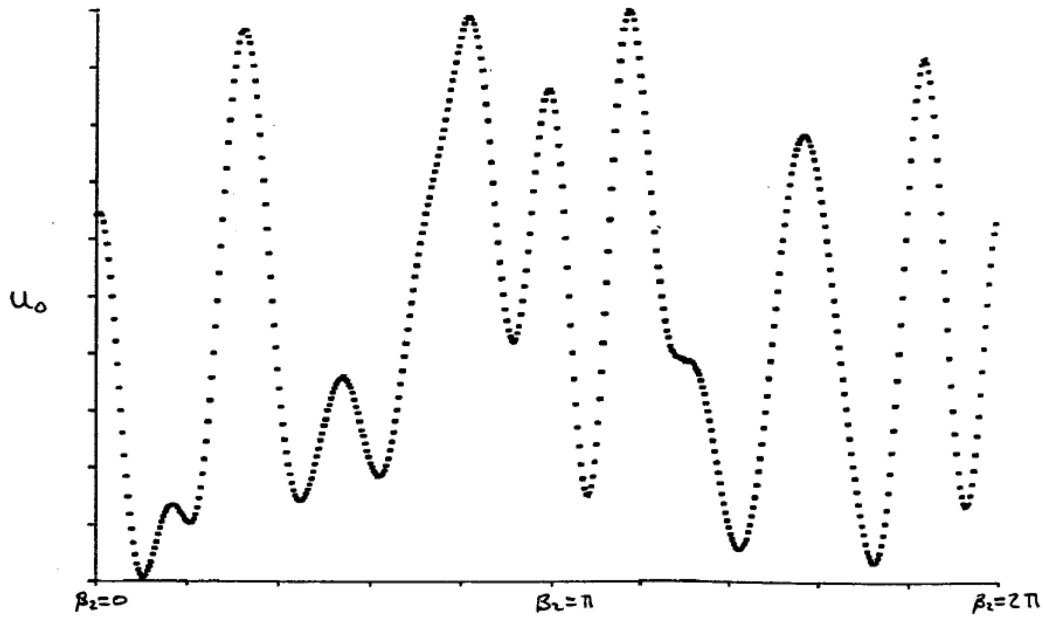


Fig. VII.9: Experimental signal corresponding to a system with three linear retarders equivalent to an elliptic retarder with parameters $\Delta = 87.2^\circ$, $\omega = 15.4^\circ$, $\Psi = 12.9^\circ$, $k = 0.919$.

If light passes through the same system in the reverse direction we obtain the following matrix

$$\mathbf{M}'_{3L} = \begin{pmatrix} 1.000 & -0.008 & -0.014 & 0.028 \\ -0.029 & 0.594 & 0.854 & -0.056 \\ 0.017 & -0.200 & 0.259 & -0.977 \\ -0.008 & -0.807 & 0.535 & 0.309 \end{pmatrix} \qquad \text{(VII.33)}$$

which approximately corresponds to an elliptic retarder given by

$$\Delta = 85.3^{\circ}, \qquad \omega = -15.9^{\circ}, \qquad \psi = 13.1^{\circ} \qquad \text{(VII.34)}$$

These results are in agreement with the reciprocity theorem T12 (in the formalism SMF). If the direction of the light through the optical medium is reversed, the obtained Mueller matrix $\mathbf{M_{3L}}$' must satisfy the relations (II.76) and (II.77) respect to $\mathbf{M_{3L}}$. Thus, as expected, the parameters Δ, ω are similar to the obtained in (VII.31), and the sign of Ψ in (VII.34) is opposite to the sign of Ψ in (VII.31) but with approximately the same value of the modulus.

According to the reciprocity theorem TR, the differences between the values (VII.35) and (VII.36) respect to the corresponding to (VII.29) and (VII.31), can be due to a lack of perpendicularity of the sample, and to an inexact reciprocity because of the internal reflections whose effect can be distinguished according to the direction of the light through the system.

VII.2.6. System composed of a polarizer and a retarder.

We have applied the Mueller polarimeter to the study of a system composed of a Polaroid HN42 linear polarizer and a Polaroid linear retarder with nominal retardation value 140 ± 20 mμ for λ = 560 nm.

For a certain orientation of the eigen-axes of the retarder and polarizer respect to the reference axes we have measured the following Mueller matrix

$$\mathbf{M}_{LP} = \begin{pmatrix} 1.000 & 0.861 & 0.426 & 0.188 \\ 0.935 & 0.816 & 0.403 & 0.201 \\ 0.315 & 0.280 & 0.156 & 0.018 \\ 0.009 & 0.001 & -0.012 & 0.008 \end{pmatrix} \qquad \text{(VII.35)}$$

from where

$$\Gamma_{\mathrm{M}} = 1.974, \qquad G_D = 0.982, \qquad G'_P = 0.985, \qquad G''_P = 0.977 \qquad \text{(VII.36)}$$

The application of the theorem T8 together with (III.26) or (III.36), gives us the following parameters for the equivalent system

$$\Delta_1 = 85.4^{\mathrm{o}}, \quad \Delta_2 = 1.7^{\mathrm{o}}, \quad \nu = 9.3^{\mathrm{o}}, \quad \xi = 21.9^{\mathrm{o}}, \quad k_{LP} = 0.033. \qquad \text{(VII.37)}$$

Taking these results to the expression (III.23) we see that the system behaves as the following

$$\mathbf{R}(\gamma)\mathbf{L}(\xi,\Delta_1)\mathbf{P}(0,k_{LP})\mathbf{R}(-\nu)\mathbf{L}(0,\Delta_2)$$

and taking into account that $\Delta_2 \approx 0,\ \gamma \approx \nu$, we see that we have a linear retarder and a linear polarizer whose axis of polarization has an angle ζ with the fast axis of the retarder, and an angle $-\nu$ with the X axis of reference.

The light beam passes first through the retarder, and then through the polarizer, and the matrix $\mathbf{M_{LP}}$ can be expressed as follows

$$\mathbf{M_{LP}} = \mathbf{M_R}(-\nu)\mathbf{M_P}(0,k_{LP})\mathbf{M_L}(\xi,\Delta_1)\mathbf{M_R}(\nu) \qquad \text{(VII.38)}$$

The intensity signal corresponding to $\mathbf{M_{LP}}$ is shown in Fig. VII.10, and it can be observed that the total period of the signal is formed by two almost equal semi periods. This fact occurs when the last element of the analyzed system is a total polarizer.

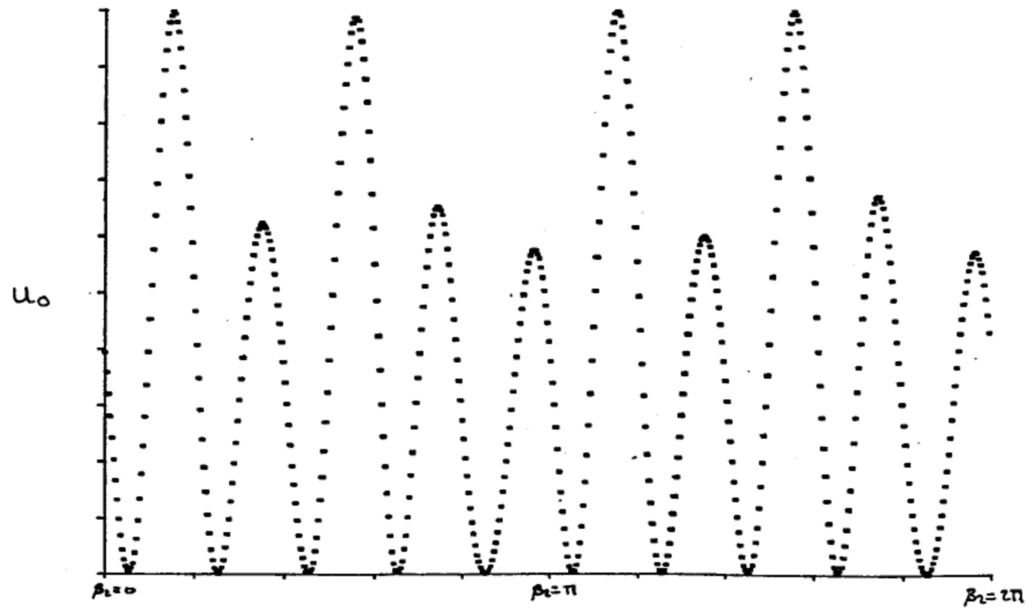


Fig. VII.10: Experimental signal corresponding to a system composed of a linear polarizer and a linear retarder categorized as L (31.2º, 85.4º) P (9.3º, 0.033).

To check the quality of the results we have made another record with an inverse order of the elements and reversing their orientations respect to the reference axis. The measured matrix is

$$\mathbf{M_{PL}} = \begin{pmatrix} 1.000 & 0.843 & 0.587 & 0.029 \\ 0.876 & 0.733 & 0.503 & 0.042 \\ 0.486 & 0.393 & 0.281 & 0.020 \\ 0.199 & 0.145 & 0.121 & 0.008 \end{pmatrix} \qquad \text{(VII.39)}$$

The application of the theorem T9 together with the expressions (III.15), takes us to an equivalent system categorized as

$\mathbf{P}(\alpha, k_{LP})\mathbf{L}(\theta,\delta)$

so that

$$\mathbf{M_{LP}} = \mathbf{M_L}(\theta,\delta)\mathbf{M_P}(\alpha, k_{LP}) \qquad \text{(VII.40)}$$

where

$$\delta = 87.0^{\circ}, \quad \theta = 32.0^{\circ}, \quad \alpha = 17.4^{\circ}, \quad k_{LP} = 0.026 \qquad \text{(VII.41)}$$

The value of δ in (VII.41) corresponds to the value of Δ_1 in (VII.37). The difference between the values is not necessarily caused by a lack of precision in the measurements, because in (VII.38) and (VII.40) we have supposed an ideal retarder.

In Fig. VII.11 we see the graphic of the intensity signal corresponding to $\mathbf{M_{PL}}$, in which is observed that the number of maximums and their positions are the same as in Fig VII.9, corresponding to $\mathbf{M_{LP}}$.

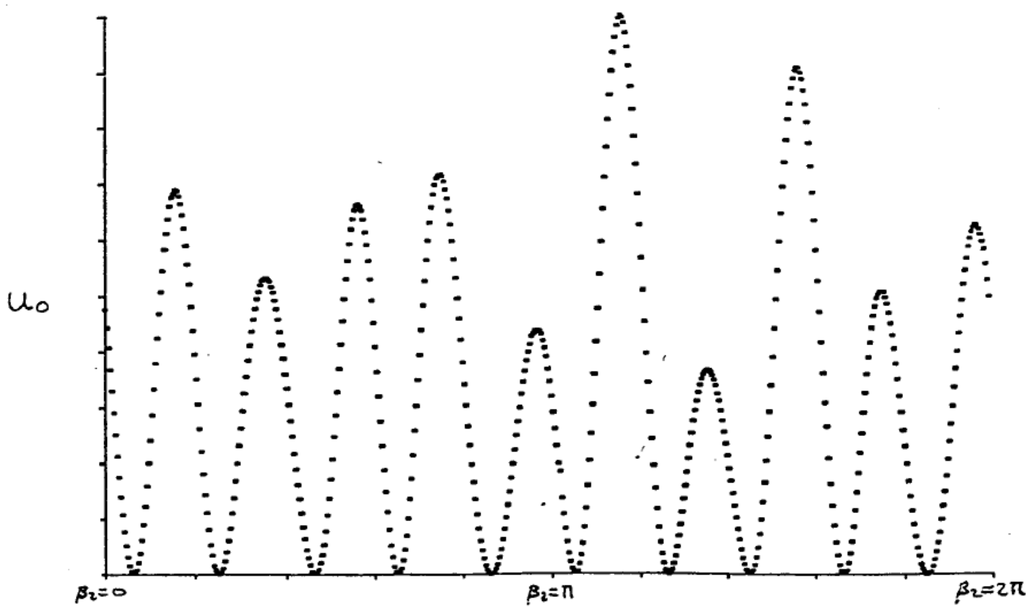


Fig. VII.11: Experimental signal corresponding to a system composed of a linear polarizer and a linear retarder categorized as P (17.4º, 0.026) L (32.0º, 87.0º).

VII.3 Discussion

Among the static methods for the analysis of polarized light we can emphasize the techniques of null ellipsometry. These techniques get a precision of the order of 0.01º in experimental measurements of angular parameters [2]. However, these techniques are not useful for the determination of Mueller matrices [23]. Some authors have proposed dynamic devices for the determination of Mueller matrices, but there are few references about the development and experimental building of such devices. In this sense it is worth mentioning the work of Thomson et al. [24], in which is described a device with four electro-optic modulators. The calibration operation of this device requires the use of several tests, with certain combinations of polarizers and retarders whose properties are previously known.

The precision cited for the measurements obtained with such device is of order of 3%.

The experimental results obtained with our device let us estimate a precision better than 1% in the values of the elements of the Mueller matrices.

Chapter VIII

Conclusions

We have developed a dynamic method for the analysis of the polarization of the light, which let us the measurement of the Stokes parameters of a given light beam and the measurement of the elements of the Mueller matrix associated with any medium active to the polarization. Both kinds of measurements are based on the Fourier analysis of the signal of light intensity supplied by the device.

The measurement of the state of polarization of a light beam is made by means of a device composed of a rotatory linear retarder and a fixed linear polarizer.

The device for the measurement of the elements of the Mueller matrices is composed of two fixed linear polarizers and two linear retarders that rotate in planes perpendicular to the direction of propagation of the light beam, with the sample medium placed between the two rotatory retarders.

The analysis of the recorded signal requires a previous self-calibration operation, which is made from the signal generated by the device in vacuum, not being necessary external calibration patterns.

We have discussed the possible values of the relation between the angular velocities of rotation of the rotatory retarders, and we have found that the most suitable value is 5/2.

To obtain the maximum physical information in the measurements of the characteristic polarimetric parameters of the material samples, we have made a theoretical study of several aspects of matricial representation, which have taken us to original contributions. Among them, we can emphasize the following:

i) The study of the restrictive relations among the elements of a Mueller matrix, which has let us to state the following theorem: "Given a Mueller matrix **M**, the necessary and sufficient condition for **M** to correspond to a non-depolarizing optical is $\mathrm{tr}\left(\mathbf{M}^T\mathbf{M}\right) = 4m_{00}^2$". We have interpreted this theorem in the Jones and Coherence Vector formalisms.

ii) We have established reciprocity theorems in the Stokes-Mueller and Coherence Vector formalisms.

iii) We have established equivalence theorems that allow the design of rotators, compensators and retardation modulators from linear retarders.

iv) Also, to know the behavior of an optical medium respect to the change in the grade of polarization, we have defined a series of parameters called Factors and Indices of Polarization and Depolarization, characteristic of the considered optical medium and obtainable from the associated Mueller matrix.

The dynamic methods for the analysis of polarized light and for the measurement of the elements of Mueller matrices introduced by us are concreted in an adequate experimental device designed and developed with the following characteristics:

i) As source of test light we use a He-Ne laser, which let us the local exploration of the samples, which is very useful in the study of inhomogeneous media.

ii) The rotatory retarders are commercial sheets, whose values of principal retardation and transmittance are not prefixed. These values are calculated during the self-calibration operation.

iii) The device has an electro-mechanic system that let us fix the origin and determine the period of the signals. Once detected by a photomultiplier and recorded by a multichannel analyzer, these signals are submitted to a computerized Fourier analysis.

In order to know the limitations and to analyze the sources of errors in the measurements obtained with our experimental device, we have made a study of several records of self-calibration and measurement of the Mueller matrices associated with several optical systems. From this, we conclude:

i) In the obtained experimental results there are systematical errors originated, we think, by the depolarization caused by the diffraction of the test light beam during the passing of the light through the several components of the device, and by the lack of perpendicularity of the surfaces of the rotatory retarders respect to the direction of propagation of the light beam.

ii) The reproducibility of the measurements and the estimated systematical errors let us be obtain an average relative error lower than 1% in the determination of the elements of Mueller matrices by means of our device.

iii) And finally, several equivalence theorems, relations among the elements of Mueller matrices, and the usefulness of the Indices of Polarization and Depolarization, have been experimentally verified with the determination of Muller matrices achieved with our device.

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