EPJ Web of Conferences **5**, 04002 (2010)
DOI:10.1051/epjconf/20100504002

Parallel decompositions of Mueller matrices and polarimetric subtraction

José J. Gil
E Universidad de Zaragoza, ICE c/ Pedro Cerbuna 12, 50009 Zaragoza, Spain

Abstract. From a general formulation of the physically realizable parallel decompositions of the Mueller matrix M of a given depolarizing system, a procedure for determining the set of pure Mueller matrices susceptible to be subtracted from M is presented. This procedure provides a way to check if a given pure Mueller matrix N can be subtracted from M or not. If this check is positive, the value of the relative cross section of the subtracted component is also determined.

1 The concept of physical Mueller matrix as an average of pure Mueller-Jones matrices

In general, an optical system can exhibit spatial heterogeneity over the area illuminated by an incident light beam, as well as dispersive effects, producing depolarization. The emerging light is consequently composed of several incoherent contributions, and the optical system cannot be represented by means of a Mueller-Jones matrix. However, taking into account that, essentially, any linear effect is due to certain sort of scattering, the system can be considered as being composed of deterministic-nondepolarizing elements, each one with a well-defined Muller-Jones matrix, in such a manner that the light beam is shared among these different elements (Fig. 1). In other words, the system is a parallel convex combination of polarimetrically pure components.

Let $I^{(i)}$ be the intensity of the portion of light that interacts with the "i" element. The ratio between $I^{(i)}$ and the intensity I of the whole beam is denoted by a respective coefficient p_i so that

$$p_i = \frac{I^{(i)}}{I}, \qquad \sum_i p_i = 1 . \tag{1}$$

Now we denote by $\mathbf{N}^{(i)}$ the Mueller-Jones matrix (also called "pure Mueller matrix") representing the "i" element. Thus, for an incident Stokes vector $\mathbf{s}$, the Stokes vector of the light pencil emerging from this element is given by

$$\mathbf{s}'_i = p_i \mathbf{N}^{(i)} \mathbf{s} . \tag{2}$$

The polarization state of the complete emerging beam is obtained through the incoherent superposition of the beams emerging from the different elements, resulting in the following Stokes vector, given by the product of the Mueller matrix of the complete system and the incident Stokes vector [1]

$$\mathbf{s}' = \sum_i \mathbf{s}'_i = \left(\sum_i p_i \mathbf{N}^{(i)} \right) \mathbf{s} = \mathbf{M}\mathbf{s}, \qquad \mathbf{M} \equiv \left(\sum_i p_i \mathbf{N}^{(i)} \right); \quad p_i \geq 0, \quad \sum_i p_i = 1 \tag{3}$$

The following sections deal with the answer to the following question: given a depolarizing Mueller matrix, how can we write it as a parallel convex combination of pure Mueller matrices?

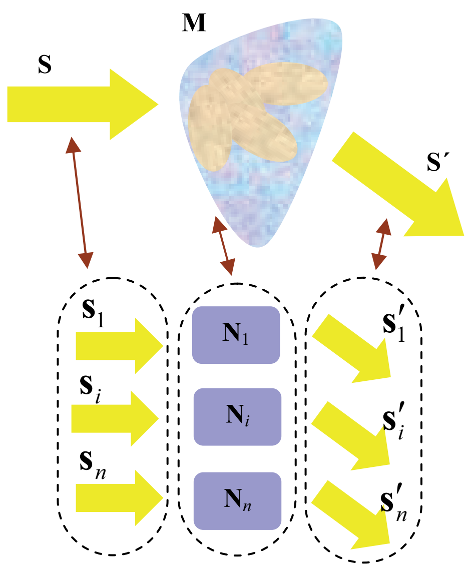

Fig. 1. A depolarizing system as a parallel composition of pure elements

2 The "spectral decomposition" of a Mueller matrix

For a given a Mueller matrix **M**, its corresponding coherency matrix **H** is [2,3]

$$\mathbf{H}(\mathbf{M})=\frac{1}{4}\begin{pmatrix} m_{00}+m_{01} & m_{02}+m_{12} & m_{20}+m_{21} & m_{22}+m_{33} \\ +m_{10}+m_{11} & +i(m_{03}+m_{13}) & -i(m_{30}+m_{31}) & +i(m_{23}-m_{32}) \\ m_{02}+m_{12} & m_{00}-m_{01} & m_{22}-m_{33} & m_{20}-m_{21} \\ -i(m_{03}+m_{13}) & +m_{10}-m_{11} & -i(m_{23}+m_{32}) & -i(m_{30}-m_{31}) \\ m_{20}+m_{21} & m_{22}-m_{33} & m_{00}+m_{01} & m_{02}-m_{12} \\ +i(m_{30}+m_{31}) & +i(m_{23}+m_{32}) & -m_{10}-m_{11} & +i(m_{03}-m_{13}) \\ m_{22}+m_{33} & m_{20}-m_{21} & m_{02}-m_{12} & m_{00}-m_{01} \\ -i(m_{23}-m_{32}) & +i(m_{30}-m_{31}) & -i(m_{03}-m_{31}) & -m_{10}+m_{11} \end{pmatrix} \quad (4)$$

where m_{ij} $(i, j = 0,1,2,3)$ are the elements of **M**.

Since **H** is a positive semidefinite Hermitian matrix [1,3], it can be diagonalized through a unitary transformation

$$\mathbf{H}=\mathbf{UD}(\lambda_0,\lambda_1,\lambda_2,\lambda_3)\mathbf{U}^+, \quad (5)$$

where $\mathbf{U}^+$ denotes the conjugate transpose of **U**, and $\mathbf{D}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ represents the diagonal matrix composed of the four non-negative eigenvalues ordered so that $0\leq\lambda_3\leq\lambda_2\leq\lambda_1\leq\lambda_0$. The columns $\mathbf{u}_i$ $(i = 0,1,2,3)$ of the 4x4 unitary matrix **U** are the respective unitary, and mutually orthogonal, eigenvectors.

Therefore, **H** can be expressed as the following convex linear combination of four rank 1 coherency matrices that represent respective pure systems [1]

$$\mathbf{H}=\frac{\lambda_0}{\mathrm{tr}\mathbf{H}}\mathbf{UD}(\mathrm{tr}\mathbf{H},0,0,0)\mathbf{U}^+ + \frac{\lambda_1}{\mathrm{tr}\mathbf{H}}\mathbf{UD}(0,\mathrm{tr}\mathbf{H},0,0)\mathbf{U}^+ + \\ +\frac{\lambda_2}{\mathrm{tr}\mathbf{H}}\mathbf{UD}(0,0,\mathrm{tr}\mathbf{H},0)\mathbf{U}^+ + \frac{\lambda_3}{\mathrm{tr}\mathbf{H}}\mathbf{UD}(0,0,0,\mathrm{tr}\mathbf{H})\mathbf{U}^+, \quad (6)$$

where each term in the sum is affected by its corresponding eigenvector $\mathbf{u}_i$, so that

$$\mathbf{H}=\sum_{i=0}^{3}\frac{\lambda_i}{\mathrm{tr}\mathbf{H}}\mathbf{H}^{(i)},\ \ \mathbf{H}^{(i)}\equiv(\mathrm{tr}\mathbf{H})(\mathbf{u}_i\otimes\mathbf{u}_i^+) \quad (7)$$

Due to the biunivocal relation between and **M**, and taking into account that $\mathrm{tr}\mathbf{H} = m_{00}$, this expression can be written in terms of the corresponding pure Mueller matrices $\mathbf{N}^{(i)}$

$$\mathbf{M}=\sum_{i=0}^{3}\frac{\lambda_i}{m_{00}}\mathbf{N}^{(i)} \quad (8)$$

This *spectral decomposition* shows that any linear system can be considered as a parallel combination of, up to four, pure systems with weights proportional to the eigenvalues of **H**. Given the physical meaning of this parallel decomposition, it is very important to keep in mind that any physically realizable parallel decomposition must be expressed as a convex linear combination. Several problems and controversies about the properties of Mueller matrices are derived from neglecting this physical requisite.

It should be noted that when an eigenvalue λ_i has a multiplicity r $(1 < r \leq 4)$, the eigenvectors of the corresponding invariant r-dimensional subspace are not unique and can be chosen arbitrarily as a set of orthonormal vectors which constitutes a basis covering the mentioned subspace.

Moreover, the statistical nature of **H** leads to a probabilistic interpretation of its eigenvalues. This fact has direct consequences in the interpretation of quantities such as the von Neumann entropy and the indices of purity [1]

3 the arbitrary decomposition of a Mueller matrix

Given the simple linear relation between a coherency matrix and its corresponding Mueller matrix, it is possible to classify the Mueller Matrices according to the rank of **H**(**M**), leading to the possible "target decompositions" [4,5,6]. The methods for decomposing measured Mueller matrices play an important role in several applications of polarimetry because they can be used for different purposes as, for example, to identify elements in the sample and to improve the contrast of images obtained by radar polarimetry [7,7].

As we have demonstrated in previous papers [8,1], the spectral decomposition is not the only possible parallel decomposition of a depolarizing system. It can also be decomposed through the "arbitrary decomposition". This decomposition can be applied to $n\times n$ coherency matrices [8,1] and, in particular, to **H**. Moreover, the physical possibility of synthesizing experimentally non-pure systems by means of different parallel combinations of pure systems shows the existence of decompositions other than the spectral decomposition.

The arbitrary decomposition of **H** into a linear convex combination of pure coherency matrices can be expressed as [1]

$$\mathbf{H}=\sum_{i=0}^{3}\frac{l_i}{\mathrm{tr}\mathbf{H}}\mathbf{A}^{(i)},\ \ \sum_{i=0}^{3}\frac{l_i}{\mathrm{tr}\mathbf{H}}=1, \\ \mathbf{A}^{(i)}=\mathbf{A}^{(i)+}=\left[(\mathrm{tr}\mathbf{H})(\mathbf{v}_i\otimes\mathbf{v}_i^+)\right],\ \ |\mathbf{v}_i|=1, \\ \mathrm{rank}\left(\mathbf{A}^{(i)}\right)=1,\quad \mathrm{tr}\mathbf{A}^{(i)}=\mathrm{tr}\mathbf{H}, \quad (9)$$

where the relative cross sections α_i of the components are given by

$$\alpha_i=\frac{l_i}{\mathrm{tr}(\mathbf{H})}, \\ l_i=\mathrm{tr}\left\{\mathbf{D}(\lambda_0,\lambda_1,\lambda_2,\lambda_3)\left[\mathbf{v}_i\otimes\mathbf{v}_i^+\right]\right\},\quad l_i>0, \quad (10)$$

and $\mathbf{v}_i$, $\mathbf{v}_j$ are linearly independent vectors that constitute a generalized basis in the subspace generated by the eigenvectors of **H** with nonzero eigenvalues. Obviously, the spectral decomposition is a particular case of the arbitrary decomposition. Each pure component can be expressed in function of the corresponding unitary vector $\mathbf{v}_i$. Moreover, in order to ensure a convex sum, the coherency matrices of the components have been chosen satisfying $\mathrm{tr}\mathbf{A}^{(i)} = \mathrm{tr}\mathbf{H} = m_{00}$. The detailed procedure for obtaining the physically realizable pure components is described in Ref. [1].

When rank (**H**) = 4, any pure system can be considered as a component and, once chosen, the successive choices of the second and third components are restricted by the exigency that the eigenvector with a nonzero eigenvalue of $\mathbf{A}^{(i)}$ belongs to the subspace generated by the eigenvectors with nonzero eigenvalues of the coherency matrix resulting from the previous subtractions. The fourth component is completely determined by the previous choices of the other pure components.

The number of pure components of the equivalent system is equal to the rank of **H**. When 1 < rank (**H**) < 4, the physically realizable pure components are restricted so that their respective characteristic vectors $\mathbf{v}_i$ lie in the respective non-null subspace [1]. When rank (**H**) = 1, the system is pure and cannot be expressed as a linear combination of different pure components.

We see that any non-pure system is polarimetrically equivalent to a parallel combination of one to four pure components with the same mean transmittance m_{00}.

The arbitrary decomposition has important consequences because it allows the identification of all the possible physically realizable target decompositions. In fact, it provides a method for analyzing measured samples and adjusting the target decomposition in order to obtain improvements in identifying unknown components. When applied to imaging polarimetry, this polarimetric subtraction can be used to improve strongly the contrast of particular elements of the target with respect to the homogeneous substrate.

The arbitrary decomposition can be expressed in terms of the corresponding Mueller matrices as follows

$$\mathbf{M} = \sum_{i=0}^{3} \alpha_i \mathbf{N}^{(i)}, \quad \alpha_i \geq 0, \quad \sum_{i=0}^{3} \alpha_i = 1, \quad n_{00}^{(i)} = m_{00} \tag{11}$$

where the Mueller matrix **M** of the system is obtained as a convex linear combination of, up to four, pure matrices $\mathbf{N}^{(i)}$ with equal mean transmittances m_{00} and respective relative cross sections given by Eq. (10).

Given a parallel decomposition, some of the pure components can be grouped into a depolarizing component. A particularly interesting example of these decompositions is the "trivial decomposition" described in Ref [1],

$$\mathbf{H} = \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{H}}\mathbf{A} + 2\frac{\lambda_1 - \lambda_2}{\mathrm{tr}\mathbf{H}}\mathbf{B}^{(1)} + 3\frac{\lambda_2 - \lambda_3}{\mathrm{tr}\mathbf{H}}\mathbf{B}^{(2)} + 4\frac{\lambda_4}{\mathrm{tr}\mathbf{H}}\mathbf{B}^{(3)}$$

$$\begin{aligned}
&\mathbf{A} \equiv \mathrm{tr}\mathbf{H}\left[\mathbf{UD}(1,0,0,0)\mathbf{U}^+\right], \qquad (12)\\
&\mathbf{B}^{(1)} \equiv \frac{1}{2}\mathrm{tr}\mathbf{H}\left[\mathbf{UD}(1,1,0,0)\mathbf{U}^+\right],\\
&\mathbf{B}^{(2)} \equiv \frac{1}{3}\mathrm{tr}\mathbf{H}\left[\mathbf{UD}(1,1,1,0)\mathbf{U}^+\right],\\
&\mathbf{B}^{(3)} \equiv \frac{1}{2}\mathrm{tr}\mathbf{H}\left[\mathbf{UD}(1,1,1,1)\mathbf{U}^+\right].
\end{aligned}$$

where $\mathbf{A}^{(0)}$ is the pure component, and $\mathbf{B}^{(i)}$ are particular non-pure components with rank $(\mathbf{B}^{(i)}) = i + 1$.

In terms of the indices of polarimetric purity [1,11]

$$\mathbf{H} = P_1\mathbf{A} + (P_2 - P_1)\mathbf{B}^{(1)} + (P_3 - P_2)\mathbf{B}^{(2)} + (1 - P_3)\mathbf{B}^{(3)}$$

4 Polarimetric subtraction of Mueller matrices

Given a Mueller matrix, the procedure described in Ref [1] for obtaining the arbitrary decomposition, gives us a method for subtracting the Mueller matrix **N** of a specific pure element, providing that it satisfies the restrictive conditions derived of the value of the rank of **H**. Thus, the pure components susceptible to be subtracted from **H** are given by respective characteristic vectors that lie in the non-null subspace of **H**.

For many experimental cases where the target under measurement contains a known (or suspected) pure component, the mentioned procedure provides a way for a proper subtraction of the known component and, thus, to isolate the unknown part of the sample. The procedure can be iterated if there exists more than one pure component to be subtracted.

An example of experimental application of the polarimetric subtraction is shown in Ref [10].

References

1. J. J. Gil, Eur. Phys. J. Appl. Phys. **40**, 1-47 (2007)
2. J. J. Gil, J. Opt. Soc. Am. A **17**, 328-334 (2000)
3. S. R. Cloude, Optik **75**, 26-36 (1986).
4. S. R. Cloude, E. Pottier, IEEE Trans. GRS **34**, 498-518 (1996)
5. F. Le Roy-Brehonnet, B. Le Jeune, Prog. Quant. Electr. **21**, 109-151 (1997)
6. F. Le Roy-Brehonnet, B Le Jeune, P Eliès, J Cariou, J Lotrian, J. Phys. D: Appl. Phys. **29**, 34-38 (1996)
7. W.-M. Boerner, in *Radar Polarimetry and Interferometry 2007* . (Educational Notes RTO-EN-SET-081bis, Paper 12. Neuilly-sur-Seine, France: RTO. pp. 12-1 – 12-30).
8. R. Touzi, W.M. Boerner, J.S. Lee, E. Lueneburg, Can. J. Remote Sensing **30**, 380–407 (2004). Available from
9. J. M. Correas, P. A.Melero, J. J. Gil, Monografías del Seminario Matemático García de Galdeano **27**, 23-240 (2003). Available from <http://www.unizar.es/galdeano/actas_pau/PDF/233.pdf>
10. M. Foldyna, E. Garcia-Caurel, R. Ossikovski, A. De Martino, and J. J. Gil, Opt. Express **17**, 12794-12806 (2009)
11. I. San Jose, J. J. Gil, arXiv:0807.2171 (2008)