Parallel decomposition of Mueller matrices
and
polarimetric subtraction

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Parallel decomposition of Mueller matrices and polarimetric subtraction

1. Concept of Mueller matrix
2. Parallel decompostions of a Mueller matrix
   - *Spectral*
   - *Trivial*
   - *Arbitrary*
3. Polarimetric subtraction

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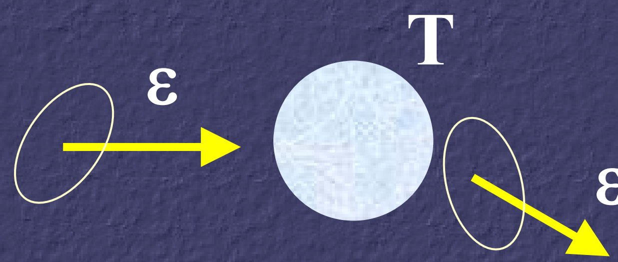
The concept of Mueller matrix

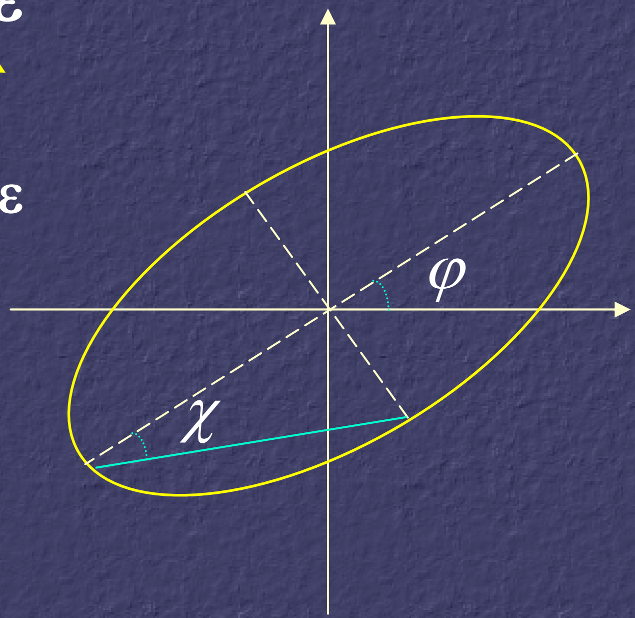
Basic interaction: Jones description



Jones vector

$$\boldsymbol{\varepsilon} = \begin{pmatrix} A_x \\ A_y e^{i\delta_y} \end{pmatrix}$$

- *Intensity I* $= \varepsilon^{+} \otimes \varepsilon$
- *DoP* $P = 1$
- χ *azimuth*
- φ *ellipticity*



$$\boldsymbol{\varepsilon}' = \mathbf{T}\,\boldsymbol{\varepsilon}$$

⬇

Jones matrix

•*Incident beam:* $P = 1$

•*Single interaction* ➡ $P' = 1$

The “pure case”

- Non-depolarizing system: for incident light whit *P*=1, emerging light has *P*’=1

- The system is equivalent to a serial combination of two components:
  - ➔ *A diattenuator (partial or total polarizer)*
  - ➔ *A retarder*

- Only 7 independent physical quantities:
  - ➔ *1 mean transmitance*
  - ➔ *3 from diattenuator*
  - ➔ *3 from retarder*

Characterization of Jones matrices

Linear passive system

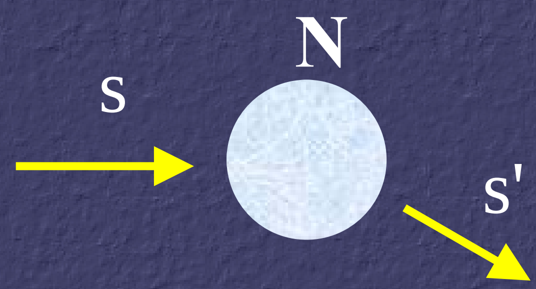
- $\mathbf{T}$ is a 2x2 complex matrix (7 physical parameters)

$$\mathbf{T} \equiv \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}, \; t_{ij} = \left| t_{ij} \right| e^{i\beta_{ij}}$$

- $\mathbf{T}$ satisfies the transmittance condition (maximum gain $\leq$ 1)

$$\frac{1}{2}\left\{ \mathrm{tr}\left(\mathbf{T}^{+}\mathbf{T}\right) + \left[\left(\mathrm{tr}\left(\mathbf{T}^{+}\mathbf{T}\right)\right)^{2} + 4\det\left(\mathbf{T}^{+}\mathbf{T}\right) \right]^{1/2} \right\} \leq 1$$

Basic interaction: Stokes-Mueller description



Stokes vector

$$\mathbf{s} \equiv \begin{bmatrix} I \\ IP\cos 2\varphi \cos 2\chi \\ IP\cos 2\varphi \sin 2\chi \\ IP\sin 2\varphi \end{bmatrix}$$

$$\mathbf{s}' = \mathbf{N}\,\mathbf{s}$$

Mueller-Jones matrix

- I *intensity*
- P *degree of polarization*
- χ *azimuth*
- φ *ellipticity*

Incident beam: $P = 1$

Single interaction → $P' = 1$

Characterization of Mueller-Jones matrices

- *7 free parameters in* $\mathbf{T} \Rightarrow$ *7 free parameters in* $\mathbf{N}$

$$\mathbf{N}\left(\mathbf{T}\right) = \mathbf{L}\left(\mathbf{T} \otimes \mathbf{T}^{+}\right)\mathbf{L}^{-1}$$

$$\mathbf{L} \equiv \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{bmatrix}$$

- *1 Transmittance condition*

$$g_f \leq 1, \quad g_f \equiv n_{00} + \underbrace{\left(n_{01}^2 + n_{02}^2 + n_{03}^2\right)^{1/2}}$$

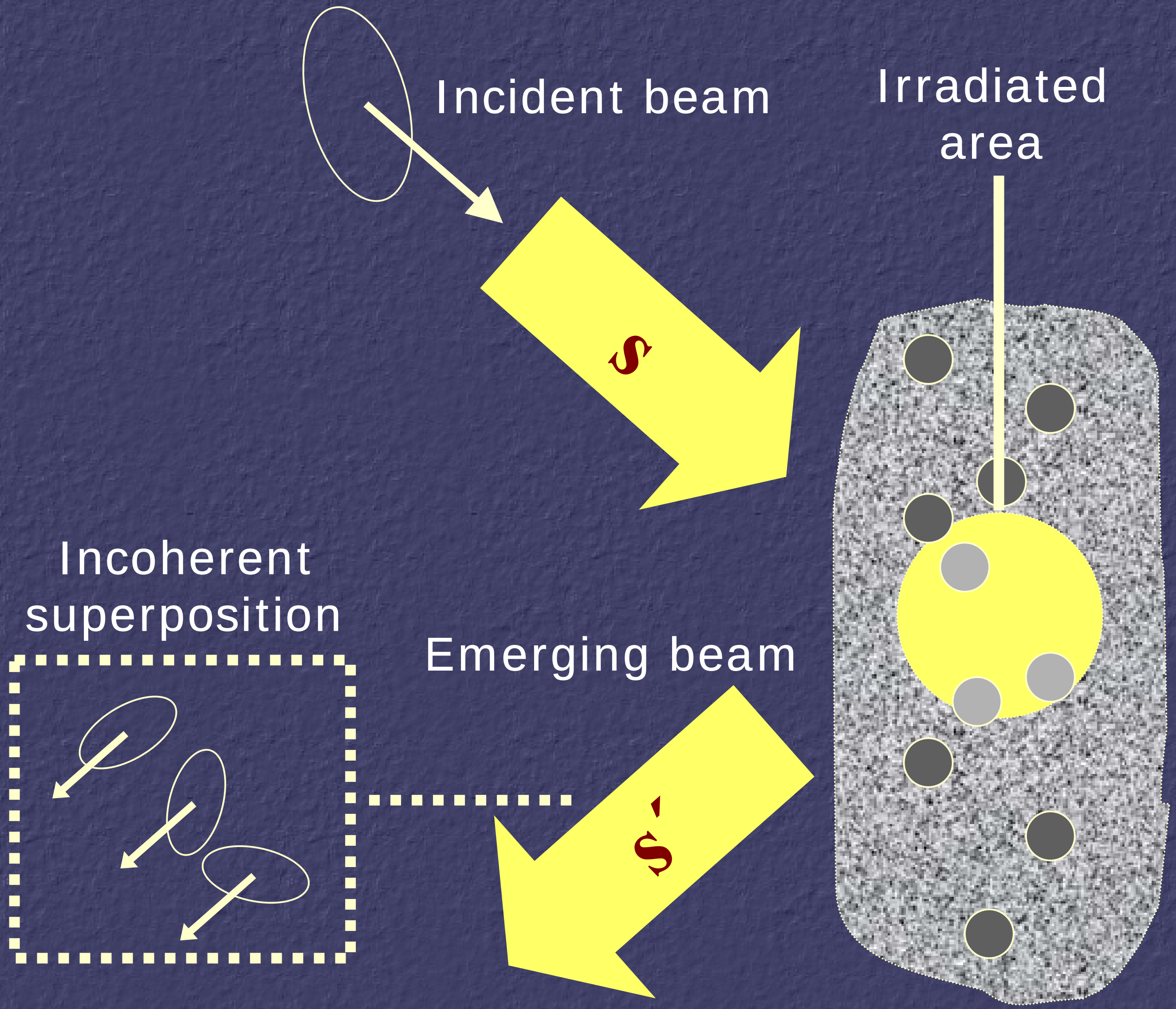
- *or*

$$g_r \leq 1, \quad g_r \equiv n_{00} + \overbrace{\left(n_{10}^2 + n_{20}^2 + n_{30}^2\right)^{1/2}}$$

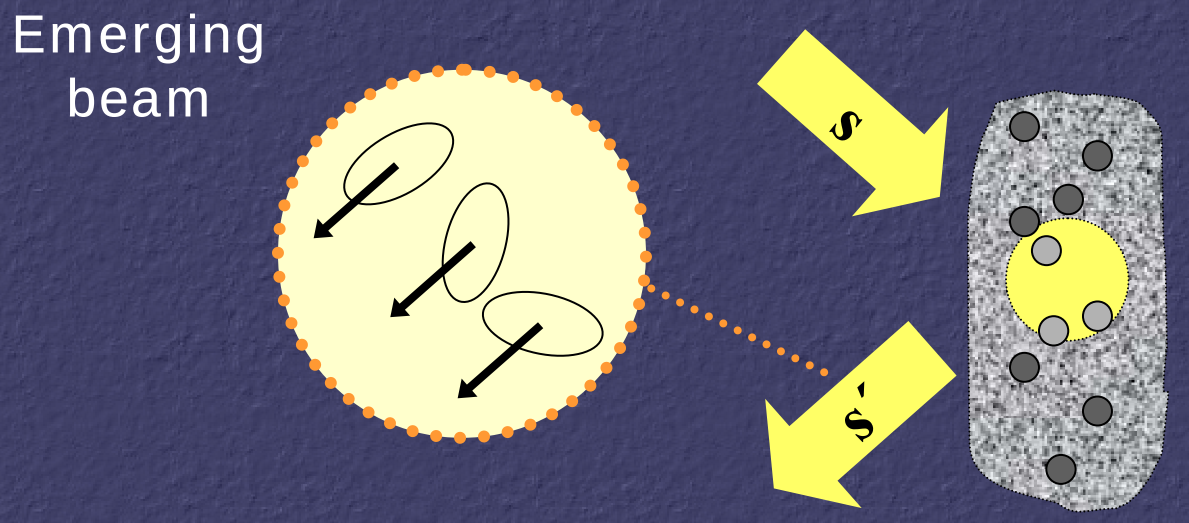
For pure systems

$g_f = g_r$

Macroscopic interaction: Synthesis of a Mueller matrix



Composed Mueller matrix



$$\mathbf{M} \equiv \left(\sum_i p_i \mathbf{N}^{(i)} \right)$$

$$\mathbf{N}^{(i)} \equiv \mathbf{L}(\mathbf{T}^{(i)} \otimes \mathbf{T}^{(i)*})\mathbf{L}^{-1}, \quad p_i \geq 0, \quad \sum_i p_i = 1$$

Coherency matrix associated with a Mueller matrix

$$\mathbf{M} \Longleftrightarrow \mathbf{H}$$

Coherency matrix $\mathbf{H}$

$\mathbf{H}$ *represents univocally the Mueller matrix and vice-versa*

$$\mathbf{H} = \frac{1}{4}\sum_{k,l=0}^{3} m_{kl}\mathbf{E}_{kl}$$

$$\mathbf{E}_{kl} = \boldsymbol{\sigma}_k \otimes \boldsymbol{\sigma}_l^*$$

$\boldsymbol{\sigma}_{kl}$ set of 4 “Pauli matrices”

$\mathbf{E}_{kl}$ set of 16 “Dirac matrices”

Coefficients m_{kl} are 16 measurable quantities: the 16 elements of the Mueller matrix $\mathbf{M}$ associated with $\mathbf{H}$

Coherency matrix $\mathbf{H}(\mathbf{M})$

$$\mathbf{H}=\frac{1}{4}\begin{pmatrix}
\begin{matrix} m_{00}+m_{01} \\ +m_{10}+m_{11} \end{matrix} & \begin{matrix} m_{02}+m_{12} \\ +i\left(m_{03}+m_{13}\right) \end{matrix} & \begin{matrix} m_{20}+m_{21} \\ -i\left(m_{30}+m_{31}\right) \end{matrix} & \begin{matrix} m_{22}+m_{33} \\ +i\left(m_{23}-m_{32}\right) \end{matrix} \\
\begin{matrix} m_{02}+m_{12} \\ -i\left(m_{03}+m_{13}\right) \end{matrix} & \begin{matrix} m_{00}-m_{01} \\ +m_{10}-m_{11} \end{matrix} & \begin{matrix} m_{22}-m_{33} \\ -i\left(m_{23}+m_{32}\right) \end{matrix} & \begin{matrix} m_{20}-m_{21} \\ -i\left(m_{30}-m_{31}\right) \end{matrix} \\
\begin{matrix} m_{20}+m_{21} \\ +i\left(m_{30}+m_{31}\right) \end{matrix} & \begin{matrix} m_{22}-m_{33} \\ +i\left(m_{23}+m_{32}\right) \end{matrix} & \begin{matrix} m_{00}+m_{01} \\ -m_{10}-m_{11} \end{matrix} & \begin{matrix} m_{02}-m_{12} \\ +i\left(m_{03}-m_{13}\right) \end{matrix} \\
\begin{matrix} m_{22}+m_{33} \\ -i\left(m_{23}-m_{32}\right) \end{matrix} & \begin{matrix} m_{20}-m_{21} \\ +i\left(m_{30}-m_{31}\right) \end{matrix} & \begin{matrix} m_{02}-m_{12} \\ -i\left(m_{03}-m_{13}\right) \end{matrix} & \begin{matrix} m_{00}-m_{01} \\ -m_{10}+m_{11} \end{matrix}
\end{pmatrix}$$

M (H)

$$\mathbf{M} = \begin{pmatrix} \begin{matrix} h_{00}+h_{11} \\ +h_{22}+h_{33} \end{matrix} & \begin{matrix} h_{00}-h_{11} \\ +h_{22}-h_{33} \end{matrix} & \begin{matrix} h_{01}+h_{10} \\ +h_{23}+h_{32} \end{matrix} & \begin{matrix} -i\left(h_{01}-h_{10}\right) \\ -i\left(h_{23}-h_{32}\right) \end{matrix} \\ \\ \begin{matrix} h_{00}+h_{11} \\ -h_{22}-h_{33} \end{matrix} & \begin{matrix} h_{00}-h_{11} \\ -h_{22}+h_{33} \end{matrix} & \begin{matrix} h_{01}+h_{10} \\ -h_{23}-h_{32} \end{matrix} & \begin{matrix} -i\left(h_{01}-h_{10}\right) \\ +i\left(h_{23}-h_{32}\right) \end{matrix} \\ \\ \begin{matrix} h_{02}+h_{20} \\ +h_{13}+h_{31} \end{matrix} & \begin{matrix} h_{02}+h_{20} \\ -h_{13}-h_{31} \end{matrix} & \begin{matrix} h_{03}+h_{30} \\ +h_{12}+h_{21} \end{matrix} & \begin{matrix} -i\left(h_{03}-h_{30}\right) \\ +i\left(h_{12}-h_{21}\right) \end{matrix} \\ \\ \begin{matrix} i\left(h_{02}-h_{20}\right) \\ +i\left(h_{13}-h_{31}\right) \end{matrix} & \begin{matrix} i\left(h_{02}-h_{20}\right) \\ -i\left(h_{13}-h_{31}\right) \end{matrix} & \begin{matrix} i\left(h_{03}-h_{30}\right) \\ -i\left(h_{12}-h_{21}\right) \end{matrix} & \begin{matrix} h_{03}+h_{30} \\ -h_{12}-h_{21} \end{matrix} \end{pmatrix}$$

Characterization of Mueller matrices

- 4 Eigenvalue Conditions

$$0 \le \lambda_i, \;\; i = 0,1,2,3$$

- 2 Transmittance Conditions

$$g_f \le 1, \;\; g_r \le 1$$

$$g_f \equiv m_{00} + \left(m_{01}^2 + m_{02}^2 + m_{03}^2 \right)^{1/2}, \quad g_r \equiv m_{00} + \left(m_{10}^2 + m_{20}^2 + m_{30}^2 \right)^{1/2}$$

Characterization theorem

A real 4x4 matrix **M** is a Mueller matrix if, and only if, the four eigenvalues of **H**(**M**) are non-negative and **M** satisfies the transmittance conditions

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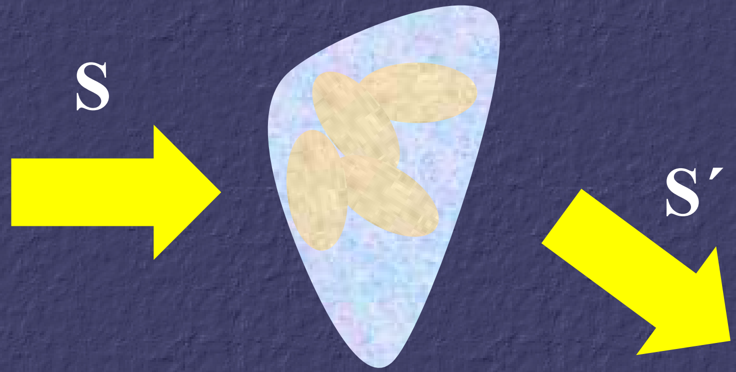
Parallel decompositions

Spectral

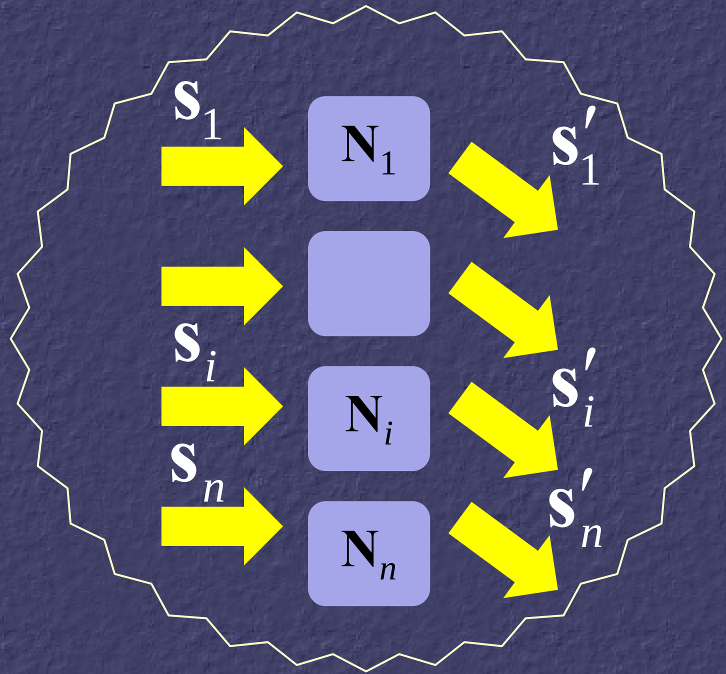
Trivial

Arbitrary

Parallel decomposition



$$\mathbf{s}' = \mathbf{M}\mathbf{s}$$



$$\mathbf{s} = \sum_{i=1}^{n} \mathbf{s}_i$$

$$\mathbf{s}'_i = \mathbf{N}_i \mathbf{s}_i$$

$$\mathbf{s}' = \sum_{i=1}^{n} \mathbf{s}'_i = \sum_{i=1}^{n} \mathbf{N}_i \mathbf{s}_i$$

Spectral decomposition

Spectral decomposition

Spectral decomposition of $\mathbf{H}$ as a convex linear combination of four systems with equal mean transmittances

$$\mathbf{H} = \mathbf{U}\underbrace{\mathbf{D}\left(\lambda_0, \lambda_1, \lambda_2, \lambda_3\right)}\mathbf{U}^+$$

$$\begin{bmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} = \overbrace{\begin{bmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}}$$

$$\mathbf{H} = \frac{\lambda_0}{\mathrm{tr}\mathbf{H}}\mathbf{U}\mathbf{D}\left(\mathrm{tr}\mathbf{H}, 0, 0, 0\right)\mathbf{U}^+ + \frac{\lambda_1}{\mathrm{tr}\mathbf{H}}\mathbf{U}\mathbf{D}\left(0, \mathrm{tr}\mathbf{H}, 0, 0\right)\mathbf{U}^+$$
$$+ \frac{\lambda_2}{\mathrm{tr}\mathbf{H}}\mathbf{U}\mathbf{D}\left(0, 0, \mathrm{tr}\mathbf{H}, 0\right)\mathbf{U}^+ + \frac{\lambda_3}{\mathrm{tr}\mathbf{H}}\mathbf{U}\mathbf{D}\left(0, 0, 0, \mathrm{tr}\mathbf{H}\right)\mathbf{U}^+$$

Spectral decomposition

$$\mathbf{H} = \frac{\lambda_0}{\mathrm{tr}\mathbf{H}}\mathbf{UD}\left(\mathrm{tr}\mathbf{H},0,0,0\right)\mathbf{U}^+ + \frac{\lambda_1}{\mathrm{tr}\mathbf{H}}\mathbf{UD}\left(0,\mathrm{tr}\mathbf{H},0,0\right)\mathbf{U}^+$$

$$+\frac{\lambda_2}{\mathrm{tr}\mathbf{H}}\mathbf{UD}\left(0,0,\mathrm{tr}\mathbf{H},0\right)\mathbf{U}^+ + \frac{\lambda_3}{\mathrm{tr}\mathbf{H}}\mathbf{UD}\left(0,0,0,\mathrm{tr}\mathbf{H}\right)\mathbf{U}^+$$

$$\mathbf{H} = \sum_{i=0}^{3} p_i \mathbf{H}_i, \quad \mathbf{N}\left(\mathbf{H}\right) = \sum_{i=0}^{3} p_i \mathbf{N}\left(\mathbf{H}_i\right)$$

each term in the sum is affected by its corresponding eigenvector $\mathbf{u}_i$

$$\mathbf{H}_i \equiv \left(\mathrm{tr}\mathbf{H}\right)\left(\mathbf{u}_i \otimes \mathbf{u}_i^+\right), \quad p_i \equiv \frac{\lambda_i}{\mathrm{tr}\mathbf{H}}, \quad \sum_{i=0}^{3} p_i = 1$$

Trivial decomposition

Trivial decomposition

$$\mathbf{H} = \mathbf{U}\underbrace{\mathbf{D}\left(\lambda_0, \lambda_1, \lambda_2, \lambda_3\right)}\mathbf{U}^+$$

$$\mathbf{D}\left(\lambda_0, \lambda_1, \lambda_2, \lambda_3\right) = \left(\lambda_0 - \lambda_1\right)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \left(\lambda_1 - \lambda_2\right)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \left(\lambda_2 - \lambda_3\right)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \lambda_3\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Trivial decomposition

Trivial decomposition of $\mathbf{H}$ as a convex linear combination of four systems with equal mean transmittances

$$\mathbf{H} = \mathbf{UD}\left(\lambda_0, \lambda_1, \lambda_2, \lambda_3\right)\mathbf{U}^+ =$$

$$= \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{H}}\mathbf{A} + 2\frac{\lambda_1 - \lambda_2}{\mathrm{tr}\mathbf{H}}\mathbf{B}_1 + 3\frac{\lambda_2 - \lambda_3}{\mathrm{tr}\mathbf{H}}\mathbf{B}_2 + 4\frac{\lambda_3}{\mathrm{tr}\mathbf{H}}\mathbf{B}_3$$

$$\mathbf{A} \equiv \mathrm{tr}\mathbf{H}\left[\mathbf{UD}\left(1,0,0,0\right)\mathbf{U}^+\right],\ \mathbf{B}_1 \equiv \frac{1}{2}\mathrm{tr}\mathbf{H}\left[\mathbf{UD}\left(1,1,0,0\right)\mathbf{U}^+\right],$$

$$\mathbf{B}_2 \equiv \frac{1}{3}\mathrm{tr}\mathbf{H}\left[\mathbf{UD}\left(1,1,1,0\right)\mathbf{U}^+\right],\ \mathbf{B}_3 \equiv \frac{1}{4}\mathrm{tr}\mathbf{H}\left[\mathbf{UD}\left(1,1,1,1\right)\mathbf{U}^+\right]$$

Trivial decomposition

Trivial decomposition of the Mueller matrix $\mathbf{M}$ as a convex linear combination of four systems with equal mean transmittances

$$\mathbf{M} = \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{H}} \mathbf{N}\left(\mathbf{A}\right)$$

$$+ 2\frac{\lambda_1 - \lambda_2}{\mathrm{tr}\mathbf{H}} \mathbf{M}_1\left(\mathbf{B}_1\right)$$

$$+ 3\frac{\lambda_2 - \lambda_3}{\mathrm{tr}\mathbf{H}} \mathbf{M}_2\left(\mathbf{B}_2\right)$$

$$+ 4\frac{\lambda_3}{\mathrm{tr}\mathbf{H}} \mathbf{M}_3\left(\mathbf{B}_3\right)$$

Indices of polarimetric purity

Dim.	**2D**	**3D**	**4D**
Coh. matrix	$\mathbf{\Phi} = \frac{1}{2}\sum_{i=0}^{3} s_i \boldsymbol{\sigma}_i$	$\mathbf{R} = \frac{1}{3}\sum_{i=0}^{8} q_i \mathbf{\Omega}_i$	$\mathbf{H} = \frac{1}{4}\sum_{i,j=0}^{3} m_{ij} \mathbf{E}_{ij}$
Purity quantities	$P = \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{\Phi}}$	$P_1 = \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{R}}$ $P_2 = \frac{\lambda_0 + \lambda_1 - 2\lambda_2}{\mathrm{tr}\mathbf{R}}$	$P_1 = \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{R}}$ $P_2 = \frac{\lambda_0 + \lambda_1 - 2\lambda_2}{\mathrm{tr}\mathbf{R}}$ $P_2 = \frac{\lambda_0 + \lambda_1 + \lambda_2 - 3\lambda_3}{\mathrm{tr}\mathbf{R}}$
Limits	$0 \le P \le 1$	$0 \le P_1 \le P_2 \le 1$	$0 \le P_1 \le P_2 \le P_3 \le 1$
Global purity	$P_{(2)} \equiv P = \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{\Phi}}$	$P_{(3)} = \frac{1}{2}\sqrt{3P_1^2 + P_2^2}$	$P_{(4)}^2 = \frac{1}{3}\left(2P_1^2 + \frac{2}{3}P_2^2 + \frac{1}{3}P_3^2\right)$

Indices of purity and trivial decomposition

Indices of polarimetric purity

$$P_1 \equiv \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{H}} \; , \; P_2 \equiv \frac{\lambda_0 + \lambda_1 - 2\lambda_2}{\mathrm{tr}\mathbf{H}} \; , \; P_3 \equiv \frac{\lambda_0 + \lambda_1 + \lambda_2 - 3\lambda_3}{\mathrm{tr}\mathbf{H}}$$

Degree of polarimetric purity

$$P_{(4)}^2 = \frac{1}{3}\left(2P_1^2 + \frac{2}{3}P_2^2 + \frac{1}{3}P_3^2 \right)$$

$$0 \le P_1 \le P_2 \le P_3 \le 1$$

- *Pure*: $P_{(4)} = P_1 = P_2 = P_3 = 1$
- *Equiprobable mixture*: $P_{(4)} = P_1 = P_2 = P_3 = 0$

Physical interpretation of the trivial decomposition

in terms of the indices of polarimetric purity

$$\mathbf{H} = \mathbf{U}\mathbf{D}\left(\lambda_0, \lambda_1, \lambda_2, \lambda_3\right)\mathbf{U}^+$$

$$= P_1\mathbf{A} + \left(P_2 - P_1\right)\mathbf{B}_1 + \left(P_3 - P_2\right)\mathbf{B}_2 + \left(1 - P_3\right)\mathbf{B}_3$$

- **A** *pure component* (rank 1)
- $\mathbf{B}_1$ *non-pure component* (rank 2)
- $\mathbf{B}_2$ *non-pure component* (rank 3)
- $\mathbf{B}_3$ *non-pure component* (rank 4), perfect depolarizer

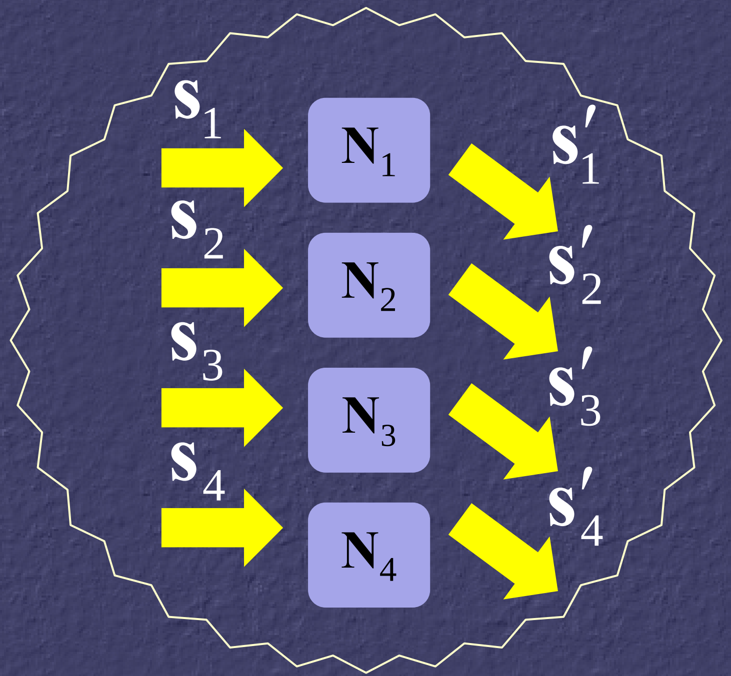
Arbitrary decomposition

Arbitrary decomposition: existence

There exist decompositions of $\mathbf{M}$ into pure components, other than the spectral decomposition?

Arbitrary decomposition

Given $\mathbf{N}_1, \mathbf{N}_2 \ldots \mathbf{N}_4$ pure elements,
we can place them as a parallel combination



$$\mathbf{s} = \sum_{i=1}^{4} \mathbf{s}_i$$

$$\mathbf{s}'_i = \mathbf{N}_i \mathbf{s}_i$$

$$\mathbf{s}' = \sum_{i=1}^{4} \mathbf{s}'_i = \sum_{i=1}^{4} \mathbf{N}_i \mathbf{s}_i$$

Arbitrary decomposition

Given $\mathbf{N}_1, \mathbf{N}_2 \ldots \mathbf{N}_4$ arbitrary pure elements, we can construct

$$\mathbf{M} \equiv \sum_{i=1}^{4} p_i \mathbf{N}_i \quad \begin{cases} \mathbf{N}_i \ \ \textit{pure Mueller matrices} \\ \sum_{i=1}^{4} p_i = 1 \end{cases}$$

Obviously, $\mathbf{N}_i$ are arbitrary, and not necessarily coincide with the “spectral” components

Arbitrary decomposition

How many “arbitrary decompositions” do exist?

Arbitrary decomposition

$$\mathbf{H} = \sum_{i=0}^{3} \frac{l_i}{m_{00}} \mathbf{A}_i \quad \begin{cases} \mathrm{tr}\mathbf{A}_i = m_{00} = \mathrm{tr}\mathbf{H} & \text{same transmittance} \\ \mathrm{rank}\left(\mathbf{A}_i\right) = 1 & \text{pure components} \\ \sum_{i=0}^{3} \frac{l_i}{m_{00}} = 1 & \text{incoherent convex sum} \end{cases}$$

$$\mathbf{H} = \sum_{i=0}^{3} \frac{l_i}{m_{00}} \left[m_{00} \left(\mathbf{v}_i \otimes \mathbf{v}_i^+ \right) \right], \quad \left| \mathbf{v}_i \right| = 1, \ \ \mathbf{v}_i \ \text{independent}$$

$$\mathbf{M} = \sum_{i=0}^{3} \frac{l_i}{m_{00}} \left[\mathbf{N}_i \right], \quad \left(\mathbf{N}_i \right)_{00} = m_{00}$$

3

Polarimetric subtraction

Statement of the problem

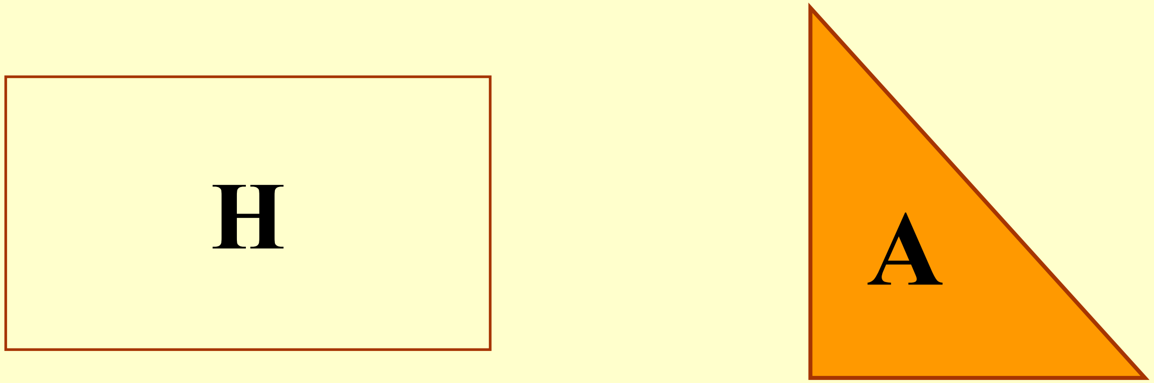
Given the coherency matrices of:

- the sample as a whole (**H**)
- a known pure component (**A**)

Find $\alpha > 0$ *such that*

$\mathbf{H}-\alpha\mathbf{A}$ is the coherency matrix of the rest

Data



Test

Is **A** compatible to be inside **H**?

Subtraction test

Subtraction is possible if,
and only if,

the only eigenvector of **A** with non-null eigenvalue

lies in the subspace generated by the eigenvectors of **H** with non-null eigenvalues

Subtraction procedure

- If $rank\,(\mathbf{H}) = 4$, the test is always positive

  Simultaneous diagonalization of $\mathbf{H}$, $\mathbf{A}$

- If $rank\,(\mathbf{H}) < 4$, and the test is positive

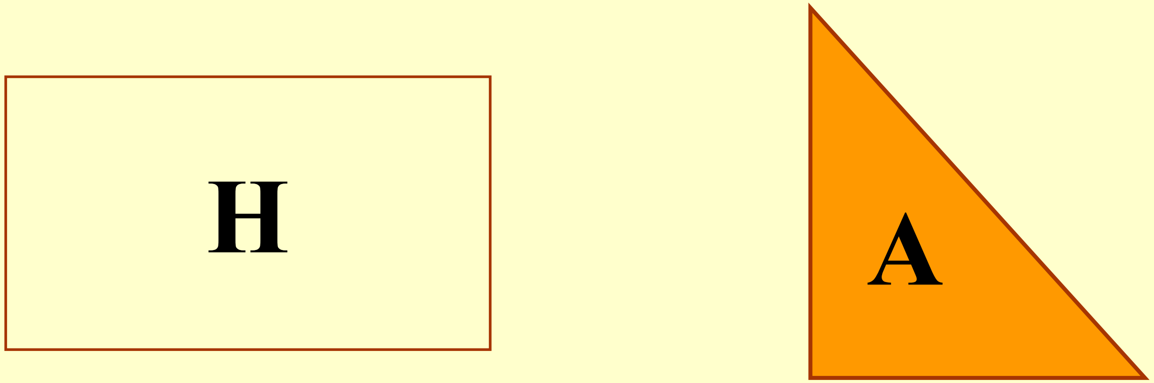
  Zeros framing of $\mathbf{H}$, $\mathbf{A}$

  Simultaneous diagonalization of $\mathbf{H}$, $\mathbf{A}$

J. J. Gil, J. M. Correas, P. A. Melero, C. Ferreira, Monogr. Semin. Mat. García Galdeano **31**, 161 (2004)

J. J. Gil, Eur. Phys. J. Appl. Phys. **40**, 1-47 (2007)

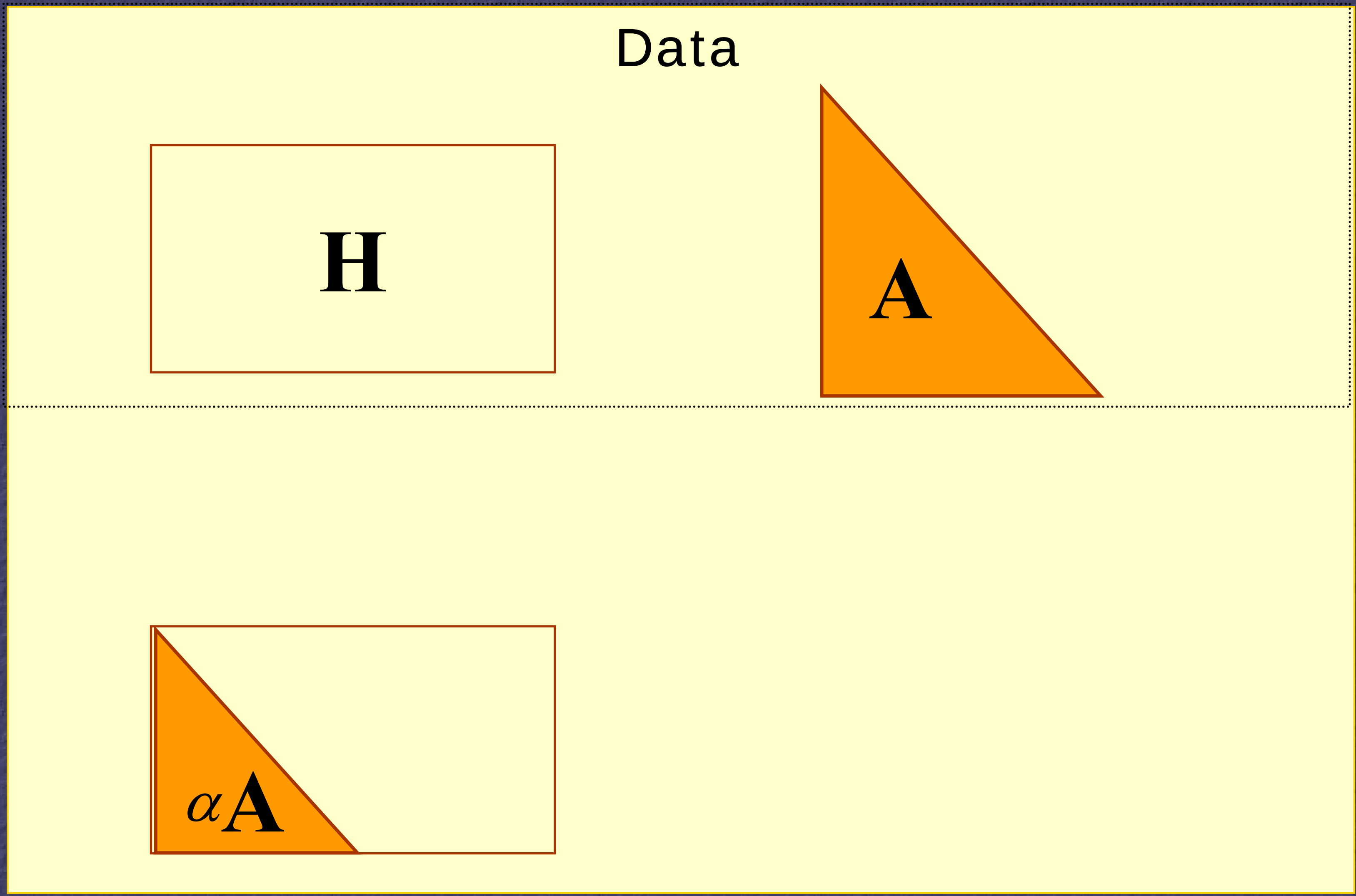
Data



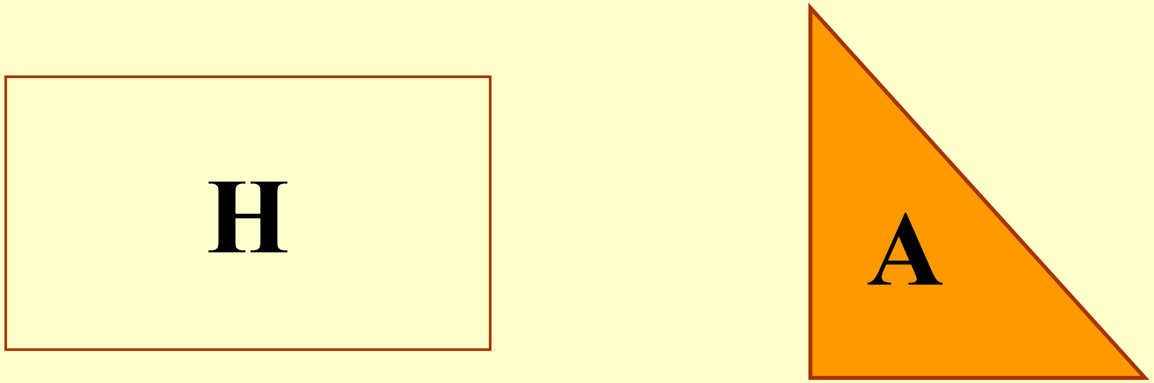
Test ? Yes

↓

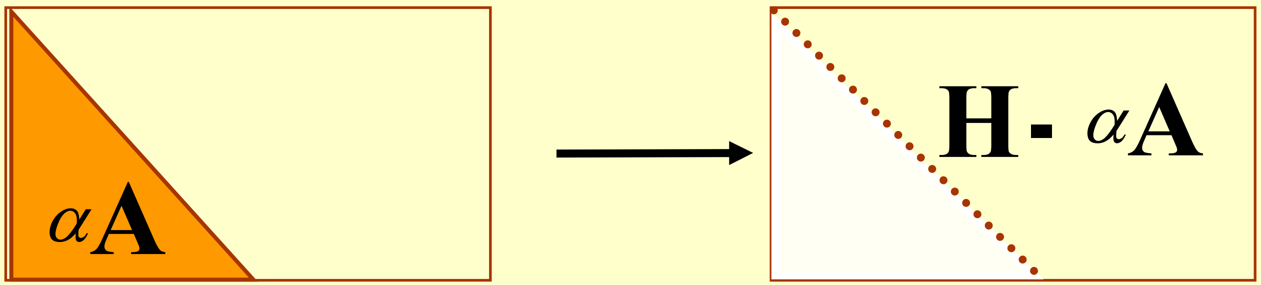
Calculate the concentration α of **A** in **H**


Data
H
A
αA

Data



Subtraction proceduure



Thank you!

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